Adventures off mass-shell

In search of a Poincaré-covariant quantum mechanics

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28 March 2014
The question

What is the correct possibility space for a relativistic quantum mechanics?

- **Space of states, defined on the mass-shell.** In this case the state space is an irreducible representation of the Poincaré group (a Wigner representation).

- **Space of states, defined on 3-momentum space.** Not Poincaré invariant, and facts about energy (and therefore mass) not representable in a state. The potential to support, given the right $\hat{H}$, many different Wigner representations—and off-mass-shell histories.

- **Space of histories, defined on 4-momentum space.** Poincaré invariant, but with many technical problems. Supports many different Wigner representations—and off mass-shell histories.
The received picture
Wigner representations
Going off mass-shell

Free space (on mass-shell)

3D space (no mass info)

4D space (off mass-shell)
Outline

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The “Hilbertized” charged scalar field

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Poincaré invariance as a reality condition

Poincaré transformations as exhaustive

Newtonian determinism

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A classical particle off mass-shell

The forced charged scalar field

Four-dimensional quantum mechanics
“Hilbertization” of a classical phase space

(See e.g. Clifton & Halvorson 2001, Geroch 2005.)

1. Begin with a classical field’s phase space $S \ni \phi$, assumed to be a vector space, equipped with a symplectic form $\sigma$, and a classical Hamiltonian $H : S \to \mathbb{R}$, which determines a Hamiltonian flow $D_t$.

2. Equip $S$ with a complex structure $J$, with which one can define $c$-number multiplication on states. $S$ is now a complex vector space.

3. With $\sigma$ and $J$, define an inner product $\langle \cdot, \cdot \rangle$, and complete $S$ in the corresponding norm, yielding a Hilbert space $S_J$.

4. We need $J$ and the quantum Hamiltonian $\hat{H}$ to be such that

$$J \dot{\phi} = \hat{H}\phi \quad (1)$$

so that $e^{-tJ\hat{H}}$ “respects” the classical Hamiltonian flow.

It follows that $\langle \phi, \hat{H}\phi \rangle = H(\phi)$, so $\hat{H}$ must have a positive spectrum.
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The classical charged scalar field – Lagrangian

- $\phi(x) : \mathbb{R}^4 \to \mathbb{C}$
- $\phi(x) : \mathbb{R}^3 \to \mathbb{C}$; $\pi_\phi(x) : \mathbb{R}^3 \to \mathbb{C}$
- Lagrangian:

  \[ L(\phi(x), \dot{\phi}(x)) = \int d^3x \ (\partial_\mu \phi^*(x) \partial_\mu \phi(x) - m^2 \phi^*(x) \phi(x)) \]  

- Define conjugate momenta

  \[ \pi_\phi(x) := \frac{\delta L}{\delta \dot{\phi}(x)} = \dot{\phi}^*(x); \quad \pi^*_\phi(x) := \frac{\delta L}{\delta \dot{\phi}^*(x)} = \dot{\phi}(x) \]

- Vary $\phi(x)$ and $\phi^*(x)$ independently.
The classical charged scalar field – Hamiltonian

- Phase space $S$ contains pairs of 3-space functions $\phi := (\phi(x), \pi_\phi(x))$.
- The Hamiltonian is given by

\[
H(\phi) := \int d^3x \left( \pi_\phi(x) \dot{\phi}(\pi_\phi(x)) + c.c. \right) - L(\phi)
\]

\[
= \int d^3x \left( |\pi_\phi(x)|^2 + |\nabla \phi(x)|^2 + m^2 |\phi(x)|^2 \right)
\]

- Note that $H$ is positive for all states $\phi \neq 0$.
- Dynamical solutions are given by the Klein-Gordon equation:

\[
\ddot{\phi}(x, t) = \nabla^2 \phi(x, t) - m^2 \phi(x, t),
\]

or

\[
\dot{\pi}_\phi^*(x, t) = -\hat{H}^2 \phi(x, t)
\]

where we introduce the differential operator $\hat{H} := \sqrt{-\nabla^2 + m^2}$. 
The classical charged scalar field – equations of motion

- Solutions expressed as a sum over “on-mass-shell” plane waves:

\[
\phi(x, t) = \int d^3k \frac{1}{\sqrt{2\omega_k}} \left( a(k)e^{i(k \cdot x - \omega_k t)} + b^*(k)e^{-i(k \cdot x - \omega_k t)} \right)
\]  (8)

where \(\omega_k := k^2 + m^2\). (The now-conventional factor \(\frac{1}{\sqrt{2\omega_k}}\) is convenient but not Poincaré-covariant.)

- The Fourier-transformed function \(\tilde{\phi}(k)\) is non-zero only on the mass-shells, where it is “Dirac delta-like”.
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The charged scalar field – positive- & negative-frequencies

- A state \( \phi = (\phi(x), \pi_\phi(x)) \) with \( H \) determines a unique history \( \phi(x, t) \).
- Performing a time-frequency Fourier transform, we determine \( \tilde{\phi}(x, \omega) \). Then define (see e.g. Wallace (2009, 13)):

\[
\phi^{(+)}(x) := \sqrt{2} \int_{0}^{\infty} d\omega \; \tilde{\phi}(x, \omega) = \int \frac{d^3k}{\sqrt{\omega_k}} a(k) e^{i k \cdot x} ; \quad (9)
\]
\[
\phi^{(-)}(x) := \sqrt{2} \int_{-\infty}^{0} d\omega \; \tilde{\phi}(x, \omega) = \int \frac{d^3k}{\sqrt{\omega_k}} b^*(k) e^{-i k \cdot x} \quad (10)
\]

- Recall \( \hat{H} := \sqrt{-\nabla^2 + m^2} \). Then

\[
\phi^{(+)}(x) := \frac{1}{\sqrt{2}} \left( \phi(x) + i \hat{H}^{-1} \pi_\phi^*(x) \right) ; \quad (11)
\]
\[
\phi^{(-)}(x) := \frac{1}{\sqrt{2}} \left( \phi(x) - i \hat{H}^{-1} \pi_\phi^*(x) \right) . \quad (12)
\]

Definable only given a full history (equivalently, \( \phi(x), \pi_\phi(x) \) and \( \hat{H} \))!
The charged scalar field – the complex structure

Define $J : S \to S$ (see Segal (1959), Segal, Whitelaw & Mackey (1963, 32)):

\[
J(\phi(x), \pi_\phi(x)) := (-\hat{H}^{-1}\pi_\phi^*(x), \hat{H}\phi^*(x))
\]  
\(13\)

\[
\Rightarrow J(\phi^+(x), \phi^-(x)) := (i\phi^+(x), -i\phi^-(x))
\]  
\(14\)

It may be checked that $J$ is a complex structure; i.e.

1. $\sigma(J\phi, J\psi) = \sigma(\phi, \psi)$, i.e. $J$ is a symplectomorphism;
2. $J^2 = -1$;
3. $\sigma(\phi, J\phi) > 0$ for $\phi \neq 0$, i.e. $\sigma(\cdot, J\cdot)$ is positive-definite.

We now have a Schrödinger equation:

\[
J\dot{\phi} = J(\dot{\phi}(x), \dot{\pi}_\phi(x)) = J\left(\frac{\delta H}{\delta \pi_\phi(x)}, -\frac{\delta H}{\delta \phi(x)}\right) = J\left(\pi_\phi^*(x), -\hat{H}^2\phi^*(x)\right)
\]  
\[
= \left(\hat{H}\phi(x), \hat{H}\pi_\phi(x)\right) = \hat{H}\phi
\]  
\(15\)
The charged scalar field – the inner product

- With $J$ and $\sigma$, we define an inner product $\langle \cdot, \cdot \rangle$ (see e.g. Clifton & Halvorson (2001, 435); Geroch (2005, p. 79)):

$$
\langle \phi, \psi \rangle := \frac{1}{2} \sigma(\phi, J\psi) + \frac{1}{2} i \sigma(\phi, \psi)
$$

(16)

$$
= i \int d^3x \left( \phi^{(+)*}(x) \frac{\partial}{\partial t} \psi^{(+)}(x) + \phi^{(-)}(x) \frac{\partial}{\partial t} \psi^{(-)*}(x) \right)
$$

(17)

$$
= \int d^3k \left( a^*(k)c(k) + b^*(k)d(k) \right).
$$

(18)

which looks (deceptively!) like the inner product for two copies of $k$-space in NRQM.

- We have $\langle \phi, \hat{H}\phi \rangle = H(\phi)$, as required.

- The spectrum of $\hat{H}$ is $[m, \infty)$, i.e. **positive**.
The charged scalar field – energy-momentum and $J$

- The choice for $J$ above allows $\hat{H}$ to be positive for both positive- and negative-frequency solutions.

- In general $P_\mu := -J \partial_\mu$, so that $\langle \phi, P^\mu \phi \rangle = T^{\mu 0}(\phi)$.

- Eigenstates of $P_\mu$ with $P^\mu = (\omega_k, k)$ have the form

$$\phi_k(x) = \sqrt{2\omega_k} a(k) e^{i(k \cdot x - \omega_k t)} + \sqrt{2\omega_k} b^*(k) e^{-i(k \cdot x - \omega_k t)}$$  \hspace{1cm} (19)

- $J$ is Poincaré-invariant for all $k^2 > 0$, so both $P_\mu$ is Poincaré covariant, and the distinction between particle and antiparticle states is Poincaré invariant.

- The definition of $J$ is tied to $\hat{H}$, and so a particular rest mass $m$.

- Therefore $S_J$ ought to be construed as the quantization of the (pair of) mass shell(s) $M_+ \cup M_-$, defined by $k^2 = m^2$. 
No position operators on the mass-shell

\[ e^{-ia\hat{Q}} \]
Why demand the single-system Hilbert space to support only one representation of the Poincaré group?

There are two possible answers, which appeal to conflicting interpretations of the PG transformations ...
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Answer 1 (e.g. Wigner (1939); Barut & Wightman (1959); Schweber (1961, 48-49)):

The objective properties of a system must be Poincaré invariant. In seeking the irreducible representations of the PG, we discover the Casimir operators, which represent these properties.

Hence seeking the irreps serves to classify the different kinds of particle that can exist in a Minkowski spacetime.

This corresponds to a \textit{passive} interpretation of the Poincaré transformations (i.e. changing one’s co-ordinate system).
Q1: Why stop at the PG?
Because the laws are Poincaré-covariant. Moving into non-inertial frames induces fictional forces.
But the laws are only Poincaré-covariant for closed systems!

Q2: Why demand irreducibility?
Because reducible reps can be constructed from these.
But by concentrating on the individual irreps to the exclusion of their direct sum, we miss all of the dynamical information—and it does not follow that the constituent systems’ state spaces each support an irrep of the PG!
E.g. $\mathcal{H}_1 \otimes \mathcal{H}_2 \supset \mathcal{H}_{\text{CoM}} \otimes \mathcal{H}_{\text{rel}}$. We expect $\mathcal{H}_{\text{CoM}}$ to support an irrep of the PG—but what about $\mathcal{H}_{\text{rel}}$?
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Answer 2:

The PG exhausts the collection of transformations one can perform on an elementary system (or a system being treated as partless). Thus action by an irrep of the PG generates a space of states for the elementary system.

This corresponds to an active interpretation of the Poincaré transformations (i.e. changing the system’s physical state—not in the sense of changing it “in world” but moving between distinct physical possibilities).
Poincaré transformations as exhaustive?

But the PG does not exhaust the transformations one can perform on an elementary system’s state!

Consider the classical particle:

- Concentrate just on tractors $q(t)$ that are analytic at $t = t_0$.
- At $t_0$ one can change location $q(t_0)$, velocity $\dot{q}(t_0)$, acceleration $\ddot{q}(t_0)$, swerve $\dddot{q}(t_0)$, ...
- We can produce any analytic-at-$t_0$ trajectory this way.
- This entails $\infty$ degrees of freedom—for a classical particle!
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- Newton’s Second Law encapsulates the deterministic principle that
  \[ \ddot{q}_i(t) = f(q_i(t), \dot{q}_i(t)) \] (20)
- This principle continues to hold for classical fields:
  \[ \ddot{\phi}(x, t) = F(\phi(x, t), \dot{\phi}(x, t)) \] (21)
  And also holds for the “Hilbertized” field (despite apparent first-order form of SE).
- Therefore states can go proxy for histories that obey the equations of motion (Wald 1994, 13).
- The PG changes the independent variables \((\phi(x), \dot{\phi}(x))\) or 
  \((\phi(x), \pi_\phi(x))\) or \((\phi^+(x), \phi^-(x))\) or \((a(k), b^*(k))\). The PG is 
  transitive in the space of dynamical solutions, given \(\hat{H}\).
Reasons to be unsatisfied

We should be looking, not in the space of *dynamical solutions* for a single system, but in the space of *kinematical possibilities*.

- **Analogy with NRQM.** States go proxy for solutions only assuming particular EoMs (due to $J$'s dependence on $\hat{H}$). We want a possibility space on which a *variety* of dynamics may be defined, as in NRQM.

- **Physical meaning.** Physical quantities ought to be meaningful independent of dynamics. And conceptually distinct quantities should be represented by distinct functions/operators on the state space. One way to have both is to define them on the space of all *kinematical* possibilities.

- **Interacting systems.** Newtonian determinism only holds for the system *as a whole*: it does not apply e.g. to either one of a pair of interacting systems. In this case one can change the higher time derivatives of the field of one system by changing the state of the other system.
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The free classical particle

(See e.g. Landau & Lifschitz (1987, 46ff.).)

\[ S = - \int m \, ds = - \int m \sqrt{1 - \dot{q}_i \dot{q}^i} \, dt = - \int \frac{m}{\gamma(\dot{q}^i)} \, dt \]  

(22)

▶ The conjugate momenta:

\[ p_i := \frac{\partial L}{\partial \dot{q}^i} = \frac{m \dot{q}^i}{\sqrt{1 - \dot{q}_i \dot{q}^i}} = \gamma(\dot{q}^i) m \dot{q}^i \]  

(23)

▶ The Hamiltonian:

\[ H(q^i, p_j) := p_i \dot{q}^i(q^j, p_j) - L(q^i, \dot{q}_i(q^j, p_j)) \]  

(24)

\[ = \sqrt{p_i p^i + m^2} \]  

(25)

so \( H^2 - p^2 = m^2 \), i.e. the particle is on-mass-shell.
The forced classical particle – electrostatic case

- Let us now impose an external electrostatic field on the particle (considered to be charged). The Lagrangian becomes

\[ L' := -\frac{m}{\gamma(\dot{q}^i)} - e\varphi(q^i) \]  

(26)

- Notice that the Lagrangian is no longer Poincaré invariant: e.g. a spatial translation will typically not preserve the value of \( \varphi(x) \)!

- The conjugate momenta are the same as before:

\[ p'_i := \frac{\partial L'}{\partial \dot{q}^i} = \frac{m\dot{q}^i}{\sqrt{1 - \dot{q}_i\dot{q}^i}} = \gamma(\dot{q}^i)m\dot{q}^i \]  

(27)

- But now the Hamiltonian:

\[ H'(q^i, p_j) := p'_i\dot{q}^i(q^i, p'_j) - L'(q^i, \dot{q}_i(q^i, p'_j)) \]  

\[ = \sqrt{p_i p^i + m^2} + e\varphi(q^i) = H + e\varphi(q^i) \]  

(29)
The forced classical particle

- Now we have
  \[ (H' - e\varphi(x))^2 - p^2 = m^2 \] (30)

- For a general electromagnetic field we have
  \[ (H' - e\varphi(x))^2 - (p' - eA(x))^2 = m^2 \] (31)

- The “mechanical” (i.e. free) Hamiltonian and momenta \((H, p)\) remain on mass-shell—by definition!

- The *physically relevant* Hamiltonian and momenta \((H', p')\) will typically be *off*-mass-shell.

- This is tantamount to: the charged particle having a different effective mass when in a non-zero field \((\Delta E = \Delta mc^2)\).

- “Physically relevant” need only imply that \(P'_\mu \mapsto -J\partial_\mu\) upon quantization.
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Four-dimensional quantum mechanics
The classical equation of motion is given by

\[
\left[ (\partial_t - ie\varphi(x))^2 - (\nabla - ieA(x))^2 \right] \phi(x) + m^2 \phi(x) = 0 \quad (32)
\]

**N.B.** This is still a linear equation, so we can still treat $S$ as a vector space.

Take the simpler, electrostatic case

\[
\left[ (\partial_t - ie\varphi(x))^2 - \nabla^2 \right] \phi(x) + m^2 \phi(x) = 0 \quad (33)
\]

Upon Hilbertization we have

\[
(J \partial_t - \hat{H})\phi(x) = iJe\varphi(x)\phi(x) \quad (34)
\]

(note the role of $iJ = \mp 1$), where $\hat{H}$ is the free Hamiltonian.
The forced charged scalar field

Assuming a weak field \((e\varphi(x) \ll m)\), this may now be calculated using Green’s functions

\[
(J\partial_t - \hat{H}_x) G(x, y) = \delta^{(4)}(x - y)
\]

(35)

where

\[
\tilde{G}(k) = \frac{1}{|k_0| - \omega_k}
\]

(36)

To first order,

\[
\langle e^{-iq \cdot x}, \phi \rangle_{4D} = \frac{1}{\sqrt{2\omega_q}} \delta(q_0 - \omega_q) a(q) + \frac{1}{\sqrt{2\omega_q}} \delta(q_0 + \omega_q) b^*(-q)
\]

\[
+ e \int_{k^2 \neq m^2} d^4k \frac{iJ\tilde{\varphi}(k)}{|k_0| - \omega_k}
\]

(37)
Consequences of going off-mass-shell

- Now there may be plane wave states with $k^2 < 0$.
- The distinction between positive-frequency states (particles) and negative-frequency state (antiparticles) is not Poincaré-invariant (Roman (1969, 119); Fraser (2008, 18)).

\[ k^\mu \quad \Lambda^\mu_\nu k^\nu \equiv -\Lambda^\mu_\nu k^\nu \]
Consequences of going off-mass-shell (2)

- $J$ is not Poincaré-invariant.
- The energy-momentum four-vector $P_\mu := -J \partial_\mu$ is not Poincaré-covariant.

- **The good news:** thanks to the $U(1)$ symmetry, the current four-vector
  \[ j_\mu(\phi)(x) := i \phi^*(x) \overleftrightarrow{\partial_\mu} \phi(x) \quad (38) \]
  is Poincaré-covariant, and $Q := \int d\sigma^\mu \ j_\mu$ is Poincaré-invariant.

- We should be prepared to accept the frame-dependence of four-momentum and particle/antiparticle-hood; all we need is a covariant charge current for a happy physical interpretation.
Conservative responses

We must accept going off-mass-shell as a consequence of being unfree.

The (unnecessarily) conservative response: avoid RQM for unfree systems (Fraser 2008).

A less conservative response: “three-dimensional” quantum mechanics:

- Consider fields in some space-like foliation.
- Then time/energy information is not contained in the state ⇒ We should not seek irreps of the PG.
- E.g. the charged scalar field as a primitively two-component field \((\phi^+(x), \phi^-(x))\), and define \(J := i\sigma_z\), independently of \(\hat{H}\).
- Accept typical violations of Poincaré-covariance for energy-momentum, and of Poincaré-invariance for the distinction between particles and antiparticles, i.e. \(\phi^+(x)\) and \(\phi^-(x)\).
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Four-dimensional quantum mechanics
The radical response: quantize *four-momentum* space.

- By a Fourier transform, this is equivalent to quantizing Minkowski spacetime.

- Related projects: Dirac (1926); Allcock (1969); Oppenheim, Reznik & Unruh (1998); Hahne (2003); Brunetti & Fredenhagen (2002); Brunetti, Fredenhagen & Hoge (2010); Olkhovsky (2011); Pashby (2013, 2014).

- Key idea: elements of the “state” space are *histories*, rather than *states* defined (non-locally!) at a time.
The general idea

- \( \mathcal{H}_{4D} = L^2(\mathbb{R}^4) \), with Poincaré-invariant inner product
  \[
  \langle \phi, \psi \rangle_{4D} := \int d^4x \phi^*(x)\psi(x).
  \] (39)

- One may now define a \textit{Poincaré-covariant} space-time “position” observables, along with four-momentum:
  \[
  (Q^\mu \phi)(x) := x^\mu \phi(x); \quad (P_\mu \phi)(x) := -i\partial_\mu \phi(x)
  \] (40)

- Dynamical solutions given by a constraint equation
  \[
  (P_0 - iJ\hat{H}[Q, P, J])\phi(x) = 0 \] (41)

Note that \(-P_0\) and \(\hat{H}\) are conceptually distinct quantities:
- \(P_0\) generates time translations;
- \(\hat{H}\) governs the dynamics (bounded from below).
- \(J\) defined independently of \(\hat{H}\), since we now have all the temporal information required to distinguish positive- and negative-frequency states.
Technical—but merely technical—problems

The spectrum of \((P_0 - iJ\hat{H})\) is continuous, so it has no eigenstates in \(L^2(\mathbb{R}^4)\). But e.g. the dynamical histories are eigenstates (eigenvalue = 0).

Possible solutions:

1. Temporal boundedness \(\Rightarrow\) non-unitary dynamics.
2. Finite universe lifetime \(\Rightarrow\) breaks Poincaré covariance.
3. “Opportunistic quantization”: choose one of \(x, y, z, t\) to be the external parameter, and quantize the remaining three dimensions (Hahne 2003).
4. Histories treated as positive linear functionals over the algebra of beables (BFH 2011).
5. Conditionalize over finite space-time hyper-volumes (Pashby 2014).
Conclusion: advantages of the 4D approach

1. The representation of states, independently of any particular $\hat{H}$.
2. ⇒ the opportunity to represent non-free dynamics.
3. ⇒ A framework for intermediate states in QFT?
4. ⇒ A response to Haag’s Theorem?
5. *Kinematical* definition of important quantities, which distinguishes e.g. the energy ($\equiv$ generator of time translations) from the Hamiltonian ($\equiv$ determiner of dynamics).
6. Poincaré-covariant position and time observables, despite Malament’s and Pauli’s Theorems. (The “non-existence” of such observables as a merely dynamical, as opposed to conceptual, restriction.)
Malament (1996, 2):

I have always taken for granted that relativity theory rules out “act-outcome” correlations across spacelike intervals. For that reason, . . . [my] result . . . does show that there is no acceptable middle ground between ordinary, non-relativistic quantum (particle) mechanics and relativistic quantum field theory.

. . .

What seems to me most important is that there is an empirical issue at stake here—whether Mother Nature does allow for act-outcome correlations across spacelike intervals. The point on which I want to stand is this: to whatever extent we have evidence that She does not allow such correlations, we have evidence that quantum mechanical phenomena must ultimately be given a field-theoretic interpretation.

Malament (1996) (see also Halvorson & Clifton (2002)): Let \( \langle \mathcal{H}, \Delta \mapsto E_\Delta, a^\mu \mapsto U(a) \rangle \) be a localization system over Minkowski spacetime (each \( \Delta \) is a spatial region of some hyperplane in a chosen foliation of spacetime) that satisfies:

1. **Localizability.** If \( \Delta \cap \Delta' = \emptyset \), then \( E_\Delta E_{\Delta'} = 0 \).
2. **Translation covariance.** \( U(a)E_\Delta U^\dagger(a) = E_{\Delta + a^\mu} \).
3. **Energy bounded below.** For any timelike \( a^\mu \), the generator of the one-parameter group \( \{ U(ta) \mid t \in \mathbb{R} \} \) has a spectrum bounded from below.
4. **Microcausality.** \( [E_\Delta, E_{\Delta + a^\mu}] = 0 \) for all spacelike \( a^\mu \) and for which the distance between \( \Delta \) and \( \Delta + a^\mu \) is nonzero.

Then \( E_\Delta = 0 \) for all \( \Delta \).
Whither Malament’s Theorem? (3)

- The projectors $E_\Delta$ do not exist for $\mathcal{H}_{4D}$ (the $\Delta$s are temporally “too thin”).

- More generally, we may deny assumption 3: *Energy bounded below.* $P_0$—the generator of time-translations—has a spectrum that is *not* bounded from below.

- What Malament’s Theorem shows is that Poincaré-covariant position operators cannot be defined on any plausible spaces of *dynamical solutions.* . . .

- . . . it does not vitiate the *meaningfulness* of such operators, which are definable on the space of *kinematical histories*.

- *Analogy:* try defining position operators for a NR system in an infinite potential well!
Borchers’ Lemma (1967) (see Malament (1996,7)):

Let \( \{ U(t) = e^{-itH} \} \) be a strongly continuous, one-parameter group of unitaries, whose generator \( H \) has a spectrum bounded from below. Let \( P_1 \) and \( P_2 \) be two projection operators such that:

- \( P_1 P_2 = 0 \); and
- there is an \( \epsilon > 0 \) s.t. for all \( t \), if \( |t| < \epsilon \),

\[
[P_1, U(t)P_2 U^\dagger(t)] = 0. \tag{42}
\]

Then \( P_1 U(t)P_2 U^\dagger(t) = 0 \) for all \( t \).
Whither Pauli’s Theorem?

- **Pauli’s Theorem:** No operator \( \hat{t} \) exists such that:
  - \( [\hat{t}, E] = -i \); where
  - \( E \) is bounded from below.

- **Strategy:** accept Pauli’s Theorem, but deny its relevance.

- Take \( E := -P_0 \equiv i\partial_t \). Then:
  - \( [\hat{t}, E] = -i \); but
  - \( E \) is *not* bounded from below.

- Take \( E := \hat{H} \). Then:
  - \( E \) is bounded from below; but
  - \( [\hat{t}, E] = -J \) on dynamical histories (\( \neq -i \)).
The PG (3D notation)

See e.g. Weinberg (1995, 61).

\[
\begin{align*}
[J_i, J_j] &= i\epsilon_{ijk} J_k , \\
[J_i, K_j] &= i\epsilon_{ijk} K_k , \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k , \\
[J_i, P_j] &= i\epsilon_{ijk} P_k , \\
[K_i, P_j] &= iH\delta_{ij} , \\
[J_i, H] &= [P_i, H] = [H, H] = 0 , \\
[K_i, H] &= iP_i .
\end{align*}
\]

(43) - (49)

What is the justification for (48), when \( H \) is an operator on states, as opposed to histories?