Individuation, entanglement and composition in permutation-invariant quantum mechanics

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Abstract

In this article I expound an understanding of the quantum mechanics of so-called “indistinguishable” systems in which permutation invariance is taken as a symmetry of a special kind, namely the result of representational redundancy. This understanding has heterodox consequences for the understanding of the states of constituent systems in an assembly and for the notion of entanglement, and corrects the inter-theoretic relations between quantum mechanics and both classical particle mechanics and quantum field theory. The most striking of the heterodox consequences are: (i) that fermionic states ought not always to be considered entangled; (ii) it is possible for two fermions or two bosons to be discerned using purely monadic quantities; and that (iii) in fact fermions (but not bosons) may always be so discerned. I conclude with a discussion of a puzzling implication for the composition of fermionic systems.

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1 Introduction

What would it mean for a quantum mechanical theory to be permutation invariant? By now the philosophy literature on permutation invariance and related issues in quantum mechanics is formidable, and a variety of construals of permutation invariance have been well articulated. The purpose of this article is to expound and advocate a construal according to which permutation invariance is treated as what I have elsewhere called an analytic symmetry. In analogy with analytic propositions, the truth of analytic symmetries—i.e. the holding of such symmetries—is a pure consequence of our choice of representational system. One surely uncontroversial example is the gauge symmetry of electromagnetic theories, which is taken by (almost\(^2\)) all to be constituted by a representational redundancy—idle wheels or “descriptive fluff” in the words of Earman (2004)—in the mathematical formalism of the theory. My motivating idea here is that permutation symmetry is constituted by a similar representation redundancy: quantum mechanics is permutation invariant because what is permuted in the mathematical formalism has no physical reality.

This doctrine, that permutation symmetry is due to representational redundancy, may be familiar, but its unassailable consequences are much less so. A central negative claim of this paper (in section 2) will be that a great deal of formal and informal discussions, in both the physics and philosophy of physics literature, tacitly rely on an interpretative doctrine that is mistaken—and even explicitly disavowed in those very same discussions.

This interpretative doctrine—which I call factorism—treats the factor Hilbert spaces of a joint Hilbert space, the latter of which represents the possible states of an assembly of systems, as having separate physical significance, even when permutation invariance has been imposed. This doctrine justifies, and is required to justify, a number of formal procedures and technical definitions. There are two particularly important examples.

1. Partial tracing. Given any state—a density operator—of an assembly of systems, the state of each constituent system is taken to be given by a reduced density operator obtained by performing a partial trace over all factor Hilbert spaces except that taken to correspond to the system of interest (see e.g. Nielsen & Chuang (2010, 105ff.)).

For the purposes of illustration, take for example the joint Hilbert space \(\mathcal{H} := \)
$\mathcal{H} \otimes \mathcal{H}$, and any density operator $\rho \in \mathcal{D}(\mathcal{H})$ in the class of unit trace, positive operators on $\mathcal{H}$. The reduced state of each two constituent system is then supposed to be given by a partial trace of $\rho$ over the other factor Hilbert space:

$$\rho_1 = \text{Tr}_2(\rho); \quad \rho_2 = \text{Tr}_1(\rho)$$

(1)

When permutation invariance is imposed, and the relevant joint Hilbert space becomes either the symmetric or anti-symmetric subspace of $\mathcal{H}$, this procedure is continued to be used to extract the states of the constituent systems. But if permutation invariance is an analytic symmetry, then factor Hilbert space labels represent nothing physical; so the prescriptions above cannot have anything other than formal significance (I will later argue that they both yield the “average state” of the constituent systems).

If that is right, then we are in need of an alternative prescription in the context of permutation invariance. I will suggest an alternative below (section 4.2). An important interpretative upshot is that the claim, commonly found in the philosophy literature, that when permutation invariance is imposed all constituent systems occupy the same state (usually an improper mixture, always so in the case of fermions), is false. In fact we will see (in section 6.2) that fermions are always in different states, as per the informal and much maligned understanding of Pauli exclusion, and that, in all but a minority of states (which in $L^2$-spaces constitute a set of measure zero), so are bosons.

The immediate consequence of this is that fermions obey a certain strong version of Leibniz’s Principle of the Identity of Indiscernibles, according to which fermions are always discernible by monadic predicates alone, a.k.a. being absolutely discernible.\(^3\) As for bosons, we will see (in section 6.3) that they too are often discernible in the same strong sense, but also that there may be utterly indiscernible—that is, not even weakly discernible—despite recent claims to the contrary (Muller and Seevinck (2009), Caulton (2013), Huggett and Norton (2013)).

2. Entanglement as non-separability. Entanglement for an assembly of systems is almost exclusively defined formally as the non-separability of the assembly’s joint state, i.e. the inability to write it as a product state in any single-system basis. If factorism is correct, then this is well justified: for then it would be true to say that a non-separable state represents a joint state of the assembly whose constituent systems do not themselves possess definite (pure) states.

But a consequence of the empty physical significance of factor Hilbert space labels is that we cannot read facts about a physical state so readily off the mathematical

\(^3\)Weyl (1928, 247) seems to have recognised this; he referred to the Pauli exclusion principle as the ‘Leibniz-Pauli principle’. Muller and Saunders (2008, 501) have claimed Weyl (1928) as an early advocate for their own view, based on a principle of charity and the belief that their account is the only one that makes fermions discernible in any non-trivial sense. But it doubtful that their subtle notion of weak discernibility is what Weyl had in mind; it is more plausible that Weyl took fermions to be discernible in the stronger sense defended here.
form of the state-vector used to represent it. Instead we should look to the algebra of admissible operators, which is greatly restricted under permutation invariance. There we find that the sort of entanglement accessible to this restricted algebra is much harder to achieve than non-separability.

I said that entanglement is almost exclusively defined as non-separability. In fact there are dissenters, chiefly Ghirardi, Marinatto & Weber (2002) (see also Ghirardi & Marinatto (2003, 2004, 2005)) (and inspired by them Ladyman et al (2013)), and Schliemann et al (2001) and Eckert et al (2002). As I will argue below (in section 5) the notion of entanglement that is most appropriate under permutation invariance agrees with the notions suggested by these authors when they agree with each other, and I use considerations from the restricted joint algebra to arbitrate between them when they disagree. A consequence of these considerations will be that fermions are not always entangled, inextricably and over cosmic distances, despite almost commonplace claims to the contrary (LeBlond, etc.).

At this point, some readers may be feeling somewhat sceptical about the current project. Interpretative philosophy of physics may seem like a recherché enterprise at the best of times, but in this case, must more ink be spilt trying to interpret a theoretical framework—elementary many-system quantum mechanics—when we already have a better theory—namely, quantum field theory (henceforth, QFT)—to which to apply our best efforts? In the state space of quantum field theory, permutation invariance is imposed at the outset, so there is no danger of being led astray by a redundant formalism. Why worry about elementary quantum mechanics?

I have two broad answers to this. (I don’t believe that they will satisfy everyone.) The first is that, despite having been superseded, elementary quantum mechanics continues to be extremely useful in solving a variety of physical problems—particularly in quantum optics and quantum information theory. Good conceptual housekeeping here means that we may be confident in physical interpretations without having first to translate everything into the language of Fock space and creation and annihilation operators. (However, such a translation can be illuminating! I offer one in section 4.4.) And, as I will argue in section 2.3, without something like the interpretation I offer here of the quantum formalism, the relation of elementary quantum mechanics to both its successor, QFT, and its predecessor, classical particle mechanics is at best obscure and at worst paradoxical.

Secondly, a better understanding of the framework of many-system elementary quantum mechanics actually aids our understanding of the framework of QFT. Not only will I advocate an interpretation of the quantum formalism that better meshes (than the current orthodoxy) with QFT in the limit of conserved total particle number, I will also argue that the metaphysical conclusions drawn in elementary quantum mechanics may be exported to QFT. Chief among these is the peculiar nature of composition for fermionic assemblies (outlined in section 7.2 and addressed in section 7).

I re-iterate that this is predominantly a project of interpretation; my goal is a better
understanding of an old theory rather than the generation of new technical results. But
interpretation and formal results cannot be sharply separated: interpretation inspires
and gives significance to formal results, and formal results offer precision in the articu-
lation of interpretations. The novel, albeit modest, results in this paper will include:

1. the articulation of a new method for extracting the states of constituent systems
   from the state of their assembly (section 4);

2. the definition of an alternative notion of entanglement for bosons and fermions,
   accompanied by a continuous measure (section 5); and

3. a proof of the fermionic counterpart to Gisin’s Theorem (1991), appropriate for
   this new notion of entanglement (section 5.2)

I should also mention—although it will already be clear—that my general interpre-
tative approach is a ‘realist’ or ‘representationalist’ one. That is, I work under the
assumption that the quantum formalism represents, or at least aims at representing,
more or less straightforwardly, underlying physical objects and processes. This sets
me apart from those who advocate a ‘pragmatic’ interpretation of quantum mechanics
(such as Healey (2012)), those who advocate a subjective or Bayesian interpretation (e.g.
Fuchs (2002)), and those who advocate not attempting an interpretation at all.

Within the representationalist camp, I also set myself apart from those who deem
the quantum formalism to be ‘incomplete’, at least at the microscopic scale; that is, I
will not consider any of the various hidden variable solutions, such as the de Broglie-
Bohm (“pilot wave”) approach. That said, my considerations can be taken as limited
to the microscopic realm, so I will not need to address the measurement problem. The
upshot is that the metaphysical claims in this article ought to be acceptable to those who
advocate either dynamical collapse, modal, or Everettian (“many world”) approaches.

Advice to the reader: Not all sections in this paper are necessary for an understanding
of my general scheme. Those sections that may safely be avoided are marked by an
asterisk (*) in their title headings.

1.1 Preliminaries: permutation invariance

I will here briefly outline the technicalities that will be in use throughout this paper. One
of the most important mathematical objects will be the single-system Hilbert space, \( \mathcal{H} \),
and its associated algebra of bounded operators, \( \mathcal{B}(\mathcal{H}) \). \( \mathcal{H} \) may be any separable Hilbert
space; therefore in the taxonomy of Murray and von Neumann (1936), \( \mathcal{B}(\mathcal{H}) \) and any
other algebra encountered in this paper will be of type I. (This will be helpful in section
3.2, where the commutativity of two single-system algebras will be taken to indicate a
tensor product structure.)

From the single-system Hilbert space \( \mathcal{H} \) we construct the \textit{prima facie} joint Hilbert
space for \( N \) equivalent systems by forming a tensor product:

\[
\mathcal{H}^{(N)} := \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H} \equiv \bigotimes_{N} \mathcal{H}
\]  

(Fraktur typeface will always be used to denote many-system Hilbert spaces). The equivalence of our constituent systems is expressed by the fact that each factor Hilbert space in (2) is a copy of the same single-system Hilbert space. Amongst other things, this equivalence means that it makes sense to speak of two constituent systems sharing the same state, so that we may talk e.g. of multiple occupation of the same single-system state.

Clearly, the equivalence of two constituent systems in this sense entails that their respective single-system Hilbert spaces are unitarily equivalent. Therefore, if these Hilbert spaces support irreducible representations of some group of spacetime symmetries (e.g. the Galilei or Poincaré group, then the systems must possess the same “intrinsic”, or state-independent, properties, such as rest mass and intrinsic spin.

The joint Hilbert space \( \mathcal{H}^{(N)} \) supports a unitary representation \( P : S_{N} \to B(\mathcal{H}^{(N)}) \) of \( S_{N} \), the group of all permutations of \( N \) symbols (for more details, see e.g. Greiner & Müller (1994, ch. 9) or Tung (1985, ch. 5)). Let \( \{|k\rangle, k \in \{1, 2, \ldots, d := \dim(\mathcal{H})\} \) be an orthonormal basis for the single-system Hilbert space \( \mathcal{H} \). Then \( P \) is defined by its action on product states of \( \mathcal{H}^{(N)} \). Let \( k : N \to d \), then for all \( \pi \in S_{N} \),

\[
P(\pi)|k(1)\rangle \otimes \ldots \otimes |k(N)\rangle := |(k \circ \pi^{-1})(1)\rangle \otimes \ldots \otimes |(k \circ \pi^{-1})(N)\rangle
\]  

The definition of \( P \) is completed by extending by linearity to the whole of \( \mathcal{H}^{(N)} \).

Permutation invariance (henceforth, PI; also known as the Indistinguishability Postulate, e.g. in French & Krause (2006, 131ff.)) is then an \( S_{N} \)-equivariance condition on the joint algebra \( B(\mathcal{H}^{(N)}) \). That is, for all \( \pi \in S_{N} \) and all \( |\psi\rangle \in \mathcal{H}^{(N)} \),

\[
\langle \psi | P^{\dagger}(\pi)QP(\pi)|\psi\rangle = \langle \psi | Q|\psi\rangle
\]  

In more physical language: PI requires that expectation values for all physical operators (operators that have a physical interpretation) be invariant under permutations of the factor Hilbert spaces. The restriction of the joint algebra from \( B(\mathcal{H}^{(N)}) \) to the algebra of operators that satisfy this condition—i.e., the \( S_{N} \)-equivariant, or symmetric operators—constitutes a superselection rule. The superselection sectors are parameterized by the irreducible representations \( \Delta_{\mu} \) (‘irreps’) of \( S_{N} \), two of which are one-dimensional: namely, the trivial representation \( \Delta_{+}(\pi) = 1 \), which corresponds to bosons, and the alternating representation \( \Delta_{-}(\pi) = (-1)^{\deg(\pi)} \) (where \( \deg(\pi) \) is the degree of the permutation \( \pi \)), which corresponds to fermions. Let us label the boson sector of \( \mathcal{H}^{(N)} \) \( \mathcal{H}^{(N)}_{+} \) and the fermion sector \( \mathcal{H}^{(N)}_{-} \). (If \( N \geq 3 \), then there are also multi-dimensional irreps, corresponding to various types of paraparticle; so \( \mathcal{H}^{(N)}_{+} \oplus \mathcal{H}^{(N)}_{-} \subset \mathcal{H}^{(N)} \), where ‘\( \subset \)’ denotes proper subspacehood. Paraparticles will not be considered for the rest of this paper.)
A note about terminology: I will talk about assemblies of “indistinguishable” systems whenever PI is imposed (I use scare quotes, because my main contention is of course that the systems are distinguishable!) and talk about “distinguishable” systems whenever PI is imposed—even if the systems are equivalent in the sense above, namely that they are individually represented by copies of the same Hilbert space.

Let me say a few words on the justification of permutation invariance. The only real justification can be that it leads to empirical adequacy, and that it does is not in question.\footnote{Cf. e.g. Prugovečki (1981, 307): ‘It has been observed that the assumption that the state $\Psi$ of the system (e.g., gas) $S$ in which particle $S_k$ is in the state $\Psi_k$ and particle $S_l$ in the state $\Psi_l$ is identical to the state of $S$ in which $S_k$ is in the state $\Psi_l$ and $S_l$ in the state $\Psi_k$ leads to a correct energy distribution in the case of particles of integer spin.’} Alternative justifications of an \textit{a priori} character are sometimes seen in the literature. One runs along the lines that, since the constituent systems of the assembly possess all the same “intrinsic” properties, one would not expect the expectation value of any beable to change upon a permutation of those systems.\footnote{Messiah & Greenberg (1964, B250), the \textit{locus classicus} for permutation invariance in quantum mechanics, write, ‘Any one of these permutations $\pi \in S_N$ is a mere reshuffling of the labels attached to the particles belonging to the same species. Since these particles are identical, it must not lead to any observable effects.’ Similar claims are made in the textbooks, e.g. Rae (1992, 205): ‘Identical particles are often referred to as \textit{indistinguishable} in order to emphasize the fact that they cannot be distinguished by any physical measurement. This implies that an operator representing any physical measurement on the system must remain unchanged if the labels assigned to the individual particles are interchanged.’} This justification is suspect for at least two reasons. First, it presumes a representational connection between the mathematical formalism and the physics which it is my main interest here to deny; namely that factor Hilbert space labels represent or denote particles (for it is the factor Hilbert space labels that are being permuted in PI).

Second, even granted this representational connection, the inference from identical “intrinsic” properties to permutation-invariant expectation values for all beables is invalid. At first blush, nothing seems particularly suspect about the beable, ‘location of system 17’ (represented by the operator $\bigotimes_{i=1}^{16} I \otimes Q \otimes \bigotimes_{i=17}^{N} I$, where $Q$ is the single-system position operator), and one would certainly \textit{not} expect this beable to preserve its expectation value upon a change of the label ‘17’ to, say, ‘5’, since then one would be talking about the location of system 5, and there is no reason to think that, just because systems 5 and 17 share the same “intrinsic” properties, it follows that they must share the same location.\footnote{Huggett and Imbo (2009) give an excellent critique of this fallacious reasoning.}

It would be a mistake to reply to this that PI does nothing but express the truism that we could have labelled system 17 e.g. with the numeral ‘5’ instead.\footnote{E.g. Merzbacher (1997, 535): ‘Since the order in which the particles are labeled has, by the definition of particle identity, no physical significance, state vectors (or wavefunctions) that differ only in the permutation of the labels must define the same state.’} PI cannot express that truism, first of all because it entails non-trivial empirical predictions, such as the divergence of assemblies in thermal states from classical Maxwell-Boltzmann statistics—how could such a non-trivial, contingent fact be explained by the truism that we could...
have chosen a different notational convention from the one that we in fact use? And
secondly, PI does not express that truism because it is reflected instead in the possibility
that, for any state-vector $|\psi\rangle$ in $S^{(N)}$, we could instead have used $P(5, 17)|\psi\rangle$ to represent
the same physical state that we currently use $|\psi\rangle$ to represent. And all that this requires
is that the two algebras associated with each labelling convention be unitarily equivalent.
But that is trivial, since $P(5, 17)\mathcal{B}(S^{(N)}) P^\dagger(5, 17)$ (where $(5, 17)$ is the permutation that
swaps symbols 5 and 17) and $\mathcal{B}(S^{(N)})$ are not only unitarily equivalent, they are identical.

There is another justification for PI that is more credible. This justification is that
factor Hilbert space labels (i.e. the order in which the factor Hilbert spaces lie in the
tensor product) represent nothing at all. If they represented nothing at all, then we
certainly would expect—indeed, we would have to ensure—that beables’ expectation
values were invariant under their permutation. Otherwise a permutation would represent
a physical difference, and that would contradict the original claim that the things being
permuted represented nothing.

In fact the only problem with this justification is that, for it to be useful as a justification,
we need already to have come to the conclusion that factor Hilbert space labels represent nothing. How do we come to that conclusion? Well, the representational
connections between our mathematical formalism and the physical world are a matter of
our choosing, but on that choice depends the experimental predictions one draws from
the theory, i.e. what we take our theory to be saying. (It might even be better to say that
theories are distinguished not only by their formalisms but also by the representational
connections that we determine for them.) And PI, or at least its consequences for the
collective behaviour of particles, is an example of those predictions. So to justify PI
in this way, we already need to know that it is obeyed—but that is just our original
justification: empirical adequacy.

While this means that we cannot take our doctrine about Hilbert space labels to
justify PI, perhaps we can reverse the situation by instead taking the empirical success
of PI as abductive support for that doctrine. This would require the absence of anything
physical, to which the factor Hilbert space labels would otherwise correspond, to be the
best explanation for PI. Is that the best explanation of PI? I see no reason to think that
the case here is any worse than the analogous case in electromagnetism, in which gauge
invariance is best explained by the physical unreality of the four-vector potential field.

Other reasons, if not to support the doctrine that factor Hilbert space labels represent
nothing, then at least to renounce the doctrine that factor Hilbert space labels represent
or denote the constituent systems of the assembly, are given in sections 2.3 and 2.4. But
first I will say a little more about that doctrine.

2 Against factorism

This section is dedicated to dispensing with the doctrine I call factorism. In section
2.1, I give a precise definition of this doctrine. In section 2.2, I contrast factorism with
haecceitism, a more familiar interpretative doctrine in the quantum philosophy litera-
ture. My criticisms of factorism comprise two sections: section 2.3 discusses the trouble
factorism causes for the inter-theoretic relation between quantum mechanics and both
classical particle mechanics and QFT; and section 2.4 argues that factorism contravenes
the principle that unitary equivalent representation represent the same physical possi-
bilities.

2.1 Factorism defined

Factorism says: particles are the physical correlates of the labels of factor Hilbert spaces.
This view is orthodox: indeed, not just orthodox, but well-nigh universal.\(^8\) It is deeply
entrenched in the way we all speak and think, and learn, about quantum mechanics for
more than one system. To explain this, and how the view is nonetheless deniable, it will
be clearest to begin by considering first, an assembly of equivalent but “distinguishable”
systems (in the sense of section 1.1). Here the relevant joint Hilbert space is \(\mathcal{H}^{(N)}\).

Each factor Hilbert space \(\mathcal{H}\) in the expression (2) for \(\mathcal{H}^{(N)}\) represents the space of
pure states for each system. The full space of states—including the mixed states—for
each particle is then represented by \(\mathcal{D}(\mathcal{H})\), the space of density operators defined on \(\mathcal{H}\).
Factorism now takes the position that the \(i\)th copy of \(\mathcal{H}\) \((\mathcal{D}(\mathcal{H}))\) represents the possible
pure (mixed) states for the \(i\)th system, so that each Hilbert space label—i.e. its position
in the tensor product in (2)—may be taken to represent or denote its corresponding
system.

I concur. I am happy to take this step: I agree that “distinguishable” systems are
the physical correlates of the labels of factor Hilbert spaces, in the usual tensor-product
formalism.

But factorism goes beyond this agreement. It says that the same goes for “indistinguis-
hable” systems: i.e. that the labels of the factor Hilbert spaces represent or denote
their corresponding systems. Or in other words: although under PI such an assembly is
described by the symmetric \((\mathcal{H}^{(N)}_+)\) or antisymmetric \((\mathcal{H}^{(N)}_-)\) subspace of the tensor prod-
uct space \(\mathcal{H}^{(N)}\), this does \textit{not} disrupt the factor spaces’ labels referring to the constituent
systems. Thus when one treats an assembly using the symmetric or antisymmetric sub-
space of \((\mathcal{H}^{(N)})\), factorism says that there are \(N\) constituent systems, one for each factor
space, and the \(i\)th system’s pure (mixed) states are represented by the state-vectors in
the \(i\)th copy of \(\mathcal{H}\) (density operators in the \(i\)th copy of \(\mathcal{D}(\mathcal{H})\)).

2.2 *Factorism and haecceitism

Some readers may be wondering how the doctrine I have called factorism differs from
haecceitism, an interpretative doctrine which is more familiar in the philosophy of quan-

\(^8\)It is a key interpretative assumption in all of the articles mentioned in footnote 1, except for Ladyman
\textit{et al} (2013).
tum theory (and logic and metaphysics quite generally). In fact they are quite different—indeed they are logically independent, as I shall argue below.

Haecceitism and its denial, anti-haecceitism, are instances of a general issue to do with the representation of possibilities. In philosophers’ jargon, the issue is whether a distinction is real, as against ‘merely verbal’, ‘spurious’ or a ‘distinction without a difference’. In the context of mathematical physics and its interpretation, the issue comes down to whether there are redundancies in the representation of physical possibilities by the mathematical objects in our theory’s formalism. That is, whether the representation relation between mathematical states (vectors or rays of the Hilbert space) and physical states is one-to-one or many-to-one.

The specific distinction with which haecceitism is associated concerns, like factorism, the action of permutations on states. There may be some mathematical states $\rho$ (which in our case are the rays or minimal projectors of the joint Hilbert space) that are wholly symmetric in the sense that their orbit under this action is a singleton set, i.e. contains only the state in question: $S_N(\rho) = \{\rho\}$. But typically a generic state $\rho$ will have a non-singleton orbit. So the question arises whether all the elements of the orbit $S_N(\rho)$ represent the same physical state of affairs.

Anti-haecceitism may be defined as always answering ‘Yes’ to this question (cf. Lewis (1986, 221)). This answer implements the intuitive idea of treating the underlying individuality of each system, the ‘which-is-which-ness’ of the systems, as physically empty or unreal. Haecceitism therefore says, on the contrary, that distinct mathematical states in an orbit represent distinct physical states. Intuitively, this implements the idea that the underlying individuality of the systems is real. However, strictly speaking if haecceitism is just the denial of anti-haecceitism, then it need not be committed to such specific claims; it need only deny the rather global claim that permuted mathematical states always correspond to the same physical state.

Haecceitism, in the sense just defined, is general enough to be assessed in classical or in quantum mechanics. Finite-dimensional classical mechanics (e.g. of $N$ point particles) is, so far as I know, almost always formulated haecceitistically, i.e. so as to distinguish states differing by a permutation of indistinguishable particles (although Belot (2001, pp. 56-61) considers the anti-haecceitistic alternative). And in infinite-dimensional classical mechanics (i.e., the mechanics of continuous media—fluids or solids), one must be a haecceitist (Butterfield (2011, pp. 358-61)).

In quantum mechanics under PI, the assessment of haecceitism is trivialized by the fact that every state of $\mathcal{H}^+(N)$ and $\mathcal{H}^-(N)$ is fixed by all permutations, i.e. $P(\pi)|\psi\rangle\langle\psi|P^\dagger(\pi) = |\psi\rangle\langle\psi|$, so we never get an orbit of permuted states larger than a singleton set. (See Pooley (2006). The situation changes when we consider paraparticles; see Caulton and Butterfield (2012b)). This already shows that factorism and haecceitism are not the same doctrine; for while the issue of haecceitism vs. its denial cannot even be articulated for “indistinguishable” systems, the issue of factorism vs. its denial can be articulated—and answered.
At this point I must raise a problem with the definition of ‘haecceitism’ that I just gave. Consider: if the precise definition of haecceitism in terms of system permutations is to be true to the metaphysical spirit of haecceitism, we must assume that the system labels must represent the objects whose which-is-which-ness the haecceitist wants to defend as real. But that assumption is just factorism. Even the standard understanding of haecceitism in quantum mechanics presumes factorism!

But a more noncommittal definition of ‘haecceitism’ is easy to formulate. We retain the emphasis on system permutations, but refrain from assuming that system permutations are represented by permutations of the factor Hilbert space. That is, we refrain from interpreting the unitary representation $P$ of $S_N$ as permuting constituent systems. (We do not give an alternative interpretation; we simply refrain from giving any interpretation.)

We therefore have two formulations of haecceitism: a general formulation, which talks of generating physical differences by a permutation of whatever represents or denotes the constituent systems, and a specific formulation, which talks of generating physical differences by a permutation of factor Hilbert spaces. Only if factorism is true are these two formulations equivalent.

We just saw that if factorism is right (and we ignore paraparticles), then the question of haecceitism cannot be settled. If, as I believe, factorism is wrong, then the specific formulation (now stripped of its standard metaphysical interpretation) remains irresolvable. There are then two remaining possibilities for the truth-value of the general formulation of haecceitism. Both are consistent, but anti-haecceitism seems to be favoured. This is because, in each of the joint Hilbert spaces $H_{+}^{(N)}$ and $H_{-}^{(N)}$, states identified by specifying occupation numbers for single-system states appear at most once. It follows that the mathematical structure required to even define a permutation of systems does not exist. Thus haecceitism (in the broad sense) could be maintained by an anti-factorist only if she is prepared to declare the standard mathematical formalism incomplete: a coherent move, but a dubious one.

In summary, factorism is a proposal about how the constituent systems of an assembly are represented in the formalism; while haecceitism, in its broad formulation, is a doctrine about how the individuality or ‘which-is-which-ness’ of those systems contributes to the individuation of joint states. Neither entails the other, but if factorism is wrong, then it is best also to consider haecceitism false.

2.3 Problems with inter-theoretic relations

Now I will turn to the first of my two criticisms of factorism, which is that it causes trouble both for obtaining a classical limit and for viewing elementary quantum mechanics as a limit of quantum field theory, when total particle number is conserved. This criticism relies on an inescapable consequence of permutation invariance and factorism: namely, that all constituent systems in an assembly possess the same state.
This can also be expressed in terms of the reduced density operators of the constituent systems. According to the usual procedure of yielding the reduced density operator of a particle by tracing out the states for all the other particles in the assembly (see e.g. Nielsen & Chuang (2010, 105)), we obtain the result that for all (anti-) symmetrized joint states, one obtains equal reduced density operators for every system. Moreover, unless the joint state is bosonic, and a product of identical factors (e.g. \( |\phi\rangle \otimes |\phi\rangle \otimes \ldots \otimes |\phi\rangle \)), then the reduced state of every system will be statistically mixed.

One consequence of the ensuing “non-individuality” of factorist systems is that they cannot become classical particles in an appropriate limit. This phenomenon is well discussed by Dieks and Lubberdink (2011), but to summarise: In the classical limit, factorist systems do not even approximately acquire the trajectories we associate with classical particles, since the former must remain in statistically mixed states all the way to the classical limit, or else possibly (if they are bosons) all remain in the same pure state. Factorist systems cannot tend, in any limit, to become distinguished one from another in space—like classical particles—by zero or at least negligible overlap, since each factorist system always possesses the “entire spatial profile” of the assembly.

A similar point can be made about quantum field theory: factorist particles do not tend to the behaviour of QFT-quanta if we consider the limit in which the total particle number in conserved. This is because QFT-quanta—which are associated with creation and annihilation operators \( a(\phi), a(\phi) \), for some state \( \phi \) in the single-particle Hilbert space \( \mathcal{H} \)—always occupy pure states. Indeed: it may be shown (though I will not here) that QFT-quanta behave just like classical particles in an appropriate classical limit for QFT. Thus the factorist systems emerge as the embarrassing odd ones out. Something has gone wrong.

2.4 Problems with unitary equivalence

My second criticism of factorism is that it defies an interpretative principle that ought to be compulsory; namely that the unitary equivalence of two Hilbert spaces and accompanying algebras is a sufficient condition for considering those Hilbert spaces to be equally good mathematical representations of the same space of physical possibilities.

Here we run into a potentially confusing ambiguity, which is an occupational hazard of doing interpretative philosophy of physics: the philosophical term ‘representation’, in the sense of a mathematical formalism representing physical facts, must be distinguished from the technical term ‘representation’, in the sense of a map from an abstract algebra to a concrete algebra of operators defined on some Hilbert space. To resolve this ambiguity, I will use the term ‘rep’ for the technical concept, and continue to use ‘representation’ for the philosophical one.

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9A selected bibliography for this result runs as follows: Margenau (1944), French & Redhead (1988), Butterfield (1993), Huggett (1999, 2003), Massimi (2001), French & Krause (2006, pp. 150-73). All these authors interpret this result as showing that Leibniz’s Principle of the Identity of Indiscernibles is pandemically false in quantum theory.
More specifically, let us define a rep of a $\ast$-algebra $\mathfrak{A}$ as an ordered pair $\langle \mathfrak{A}, \pi \rangle$ of a Hilbert space $\mathcal{H}$ and a $\ast$-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathfrak{H})$. Then two reps $\langle \mathfrak{A}, \pi \rangle$ and $\langle \mathfrak{B}, \phi \rangle$ are **unitarily equivalent** iff there is a unitary operator $U : \mathfrak{H} \to \mathfrak{K}$ such that $U\pi[\mathfrak{A}] = \phi[\mathfrak{A}]U$ (i.e. there is an **intertwiner** $U$ between $\pi[\mathfrak{A}]$ and $\phi[\mathfrak{A}]$).

I emphasise that the unitary equivalence of two Hilbert spaces is a much stronger condition than their being isomorphic (which requires only that the have the same dimension). Unitary equivalence means that the same abstract algebra of operators is being realised by the concrete algebras of two Hilbert spaces in such a way that preserves **all** expectation values. It seems uncontroversial, then, that we should consider any two unitarily equivalent reps to represent the same space of physical states, and furthermore we should consider the intertwiner $U$ to preserve physical interpretations.

However, there is a wrinkle here: not every physically relevant quantity may be represented by the abstract algebra. If mathematical artefacts of one rep but not another are also doing representational work, then we have a reason to block the inference from unitary equivalence by the intertwiner $U$ to preserved physical interpretation under action by $U$.

For example, consider the joint Hilbert space $\mathfrak{K}^{(2)} := \mathcal{H} \otimes \mathcal{H}$, and suppose that the two systems they describe are distinguished one from the other by at least one "intrinsic" or state-independent property $F$ (perhaps a haecceity) that one of the systems—but not the other—possesses. Suppose that it is the first copy of $\mathcal{H}$ in the tensor product that represents the system with $F$. How could $F$ be represented in the mathematical formalism? Certainly not by any non-trivial operator on $\mathcal{H}$, since the property is state-independent. But not by a trivial operator—some multiple of the identity, $\lambda I$, where $\lambda \in \mathbb{C}$—on $\mathcal{H}$ either, since both systems are represented by that Hilbert space, and it would yield the same value both times. Similar considerations lead to the conclusion that there is no way to represent $F$ on $\mathfrak{K}^{(2)}$ either.

The only solution is to treat one of the factor Hilbert spaces as distinguished, as it were “outside” the formalism. But the honour of being distinguished in this way will not be preserved under unitary equivalence. For example, select some subspace $S_1$ of the first copy of $\mathcal{H}$ (the copy that represents the system with $F$) and some subspace $S_2$ of the second copy. Then the joint Hilbert space $\mathfrak{K}_s := S_1 \otimes S_2$ represents a subspace of the joint Hilbert space $\mathfrak{K}^{(2)}$. But $\mathfrak{K}_s$, with its associated algebra, is unitarily equivalent to $\mathfrak{K}_p := S_2 \otimes S_1$ ("p" for ‘permute’) and its associated algebra: the intertwiner is the permutation operator $P(12)$ restricted to $\mathfrak{K}_s$. And while it is true that $\mathfrak{K}_p$ would represent equally well the physical states currently represented by $\mathfrak{K}_s$, it is not true that $P(12)$ preserves the physical interpretation of the states between $\mathfrak{K}_s$ and $\mathfrak{K}_p$. For example, the state $|\Psi\rangle := |\phi\rangle \otimes |\chi\rangle$ represents the physical state in which the system with $F$ is in the state (represented by) $|\phi\rangle$ and the system without $F$ is in the state (represented by) $|\chi\rangle$. But the permuted state $P(12)|\Psi\rangle$ instead represents the physical state in which the system with $F$ is in the state (represented by) $|\chi\rangle$ and the system without $F$ is in the state (represented by) $|\phi\rangle$. In summary, so long as there are deemed to be **any** physical quantities not represented in the relevant algebras, we cannot allow that the
intertwiner between two unitary equivalent reps preserves the physical interpretation of the components of those two reps.

The remedy for this wrinkle is simply to add a invariance condition to our interpretative principle:

If two reps are unitarily equivalent, and any physical quantity not represented in either algebra is invariant between them, then the intertwiner between the two reps preserves the physical interpretation of any component of those two reps.

This deals with the problem above, since the identity of the system that possesses $F$ is not invariant between $H_s$ and $H_p$.

Factorism runs afoul of this interpretative principle. For, as the results of the next section (3.2) show, there are subspaces of the symmetric and anti-symmetric spaces $\mathcal{H}^{(N)}_+$ and $\mathcal{H}^{(N)}_-$ that are unitarily equivalent to subspaces of $\mathcal{H}^{(N)}$. And so long as we take the order of the factor Hilbert spaces to have no physical significance (for this is not invariant between the subspaces), we ought, by the principle above, to give the same physical interpretation to these subspaces. But that requires denying the result mentioned above in section 2.3 that, under permutation invariance, all constituent systems are in the same physical state. And the only way to resist that result is to relinquish the interpretative doctrine that justifies taking the partial trace of a joint state to obtain the reduced state of its constituents, which is factorism.

3 Qualitative individuation

This section inaugurates the positive part of this paper. The foregoing criticisms of factorism have left us in need of an alternative procedure for extracting the states of constituent systems from the joint state of an assembly. What we need is some way to decompose the joint Hilbert space ($\mathcal{H}^{(N)}_+$ or $\mathcal{H}^{(N)}_-$) in a way that obeys the strictest interpretation of permutation invariance.

3.1 Decomposing the right joint Hilbert space

One might say that factorism fails by trying to decompose the wrong joint Hilbert space. Rather than decomposing $\mathcal{H}^{(N)}_+$ or $\mathcal{H}^{(N)}_-$ into single-system Hilbert spaces, it instead forgets the superselection rule induced by permutation invariance, embeds $\mathcal{H}^{(N)}_+$ and $\mathcal{H}^{(N)}_-$ back into $\mathcal{H}^{(N)}$, and proceeds to decompose that instead. And the decomposition of this space is easy, since by construction it has a tensor product structure. However, what we want to do is take the superselection rule seriously, and try to decompose $\mathcal{H}^{(N)}_+$ or $\mathcal{H}^{(N)}_-$.
A natural idea is to try to find a similar $N$-fold tensor product structure in $\mathcal{H}_+^{(N)}$ or $\mathcal{H}_-^{(N)}$. In more detail, this would entail finding $N$ putatively single-system Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$, with associated single-system algebras, such that the tensor product Hilbert space formed from these, $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N$, is unitarily equivalent to one of $\mathcal{H}_+^{(N)}$ or $\mathcal{H}_-^{(N)}$.

But we face an immediate problem: these joint spaces may well have a **prime** number of dimensions,\(^\text{10}\) which entails that only one of the would-be factor Hilbert spaces $\mathcal{H}_k$ could have more than 1 dimension! Worse: it is hard to see what physical interpretation, in terms of single-system states, one could give to the multi-dimensional factor space $\mathcal{H}_k$.

However, this doomed idea can be rehabilitated: for we need not be so ambitious so as to decompose the **whole** of the joint Hilbert space in one go. Instead, we can try decomposing **subspaces** of $\mathcal{H}_+^{(N)}$ or $\mathcal{H}_-^{(N)}$.

Suppose such a decomposition successful in principle for some subspace $\mathcal{S}$. Then the collection of constituents corresponding to the decomposition must be interpreted as co-existing only in those states belonging to $\mathcal{S}$. This is not objectionable **per se**. Agreed: in the case of “distinguishable systems” there are means of individuating systems which will suffice for all states. But if one is not a haecceitist, why should one demand or expect this all the time?

If there are two subspaces, $\mathcal{S}_1$ and $\mathcal{S}_2$ say, each of which may be decomposed, then the question arises whether any constituent system represented in $\mathcal{S}_1$ is the same as any constituent system represented in $\mathcal{S}_2$. It will turn out that this question does not have an unequivocal answer. The associated metaphysical picture is one in which relations of “trans-state identity” (as we might call it) have no objective significance, a quantum analogue of Lewis’s (1968) celebrated Counterpart Theory. We should not be surprised: as we saw in section 2.2, the anti-factorist picture, like Lewis’s, eschews haecceitism.

Now that we have limited our search for decompositions to subspaces of the joint Hilbert space, it remains to be shown that such subspaces exist. It is the purpose of the next section to prove that they do.

### 3.2 Natural decompositions

I will use the term **individuation** for the act of picking out an object, or collection of objects, according to some property that it may have; and I will call the property in question the **individuation criterion**. Since individuation in this sense need not entail uniqueness, an individuation criterion is not quite a Russellian definite description.

I will also say that an object, or class of objects, is **qualitatively** individuated iff its individuation criterion is a qualitative property. I will assume that qualitative properties are represented by projectors in the single-system Hilbert space $\mathcal{H}$. (Factorist

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\(^{10}\)For example, if $\mathcal{H} \cong \mathbb{C}^2$, then $\mathcal{H}_+^{(2)} \cong \mathbb{C}^3$. 

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individuation—i.e. individuation according to factor Hilbert space labels—may be considered non-qualitative individuation.) These projectors need not be minimal, i.e. 1-dimensional. But a minimal projector corresponds to a maximally logically strong qualitative property (something like a Leibnizian ‘individual concept’).

Since I deny factorism, in the context of PI qualitative individuation is our only means of picking out constituent systems. I will argue here that the decomposable subspaces of the joint Hilbert space may correspond to physical states in which the constituent systems have been qualitatively individuated.

What counts as a successful decomposition of a joint Hilbert space? What are our criteria for success? Here I draw upon the work of Zanardi (2001) and Zanardi et al (2004), which emphasises working in terms of algebras of beables. Zanardi (2001, p. 3) writes:

When is it legitimate to consider a pair of observable algebras as describing a bipartite quantum system? Suppose that \( A_1 \) and \( A_2 \) are two commuting \(*\)-subalgebras of \( A := \text{End}(\mathcal{H}) \) such that the subalgebra \( A_1 \vee A_2 \) they generate, i.e., the minimal \(*\)-subalgebra containing both \( A_1 \) and \( A_2 \), amounts to the whole \( A \), and moreover one has the (noncanonical) algebra isomorphism,

\[
A_1 \vee A_2 \cong A_1 \otimes A_2
\]  

The standard, genuinely bipartite, situation is of course \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, A_1 = \text{End}(\mathcal{H}_1) \otimes \mathbf{1}, \) and \( A_2 = \mathbf{1} \otimes \text{End}(\mathcal{H}_2) \). If \( A_1' := \{ X \mid [X, A_1] = 0 \} \) denotes the commutant of \( A_1 \), in this case one has \( A_1' = A_2 \).

Thus Zanardi’s proposal is to work by analogy with the case of distinguishable systems: we look for commuting subalgebras whose tensor product is isomorphic to the joint (symmetric) algebra for the assembly’s Hilbert space. The ‘(noncanonical) algebra isomorphism’ can for us be unitary equivalence.

Zanardi et al (2004, p. 1) offer three necessary and jointly sufficient conditions for what they coin a natural decomposition. I will express these by setting \( N = 2 \), for definiteness (so that we are seeking a decomposition into two constituent systems).

- **Local accessibility.** This condition states that the subalgebras be ‘controllable’, i.e. experimentally implementable. We have no need of this condition here, since our interest is not experimental but metaphysical. But we may replace it with the condition that the subalgebras have a natural physical interpretation as single-system algebras. This means that the beables in these algebras ought to have a recognizable form as monadic quantities, such as position, momentum and spin.

- **Subsystem independence.** This condition requires the subalgebras to commute:

\[
\forall A \in A_1, \forall B \in A_2 : [A, B] = 0. \]  

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I.e., each system ought to possess its properties independently of the other. This condition will be familiar from algebraic quantum field theory, where it is imposed on observable algebras associated with space-like separated, compact spacetime regions under the name microcausality, and interpreted as vetoing the possibility of act-outcome correlations between space-like separated events. (For more details, see Halvorson (2007, sections 2.1, 7.2).)

- Completeness. This condition requires the minimum algebra containing both sub-algebras to amount to the original joint algebra $A$:

$$A = A_1 \vee A_2$$  \hspace{1cm} (7)

This expresses the fact that the assembly in question has been decomposed without residue.

We are here dealing only with separable Hilbert spaces, and therefore, in the taxonomy of Murray and von Neumann (1936), with type I algebras; it follows that the second two conditions entail the tensor product structure expressed above in (5). And we read the isomorphism ‘$\cong$’ as a claim of unitary equivalence. Thus we can say our joint Hilbert space and its associated algebra has been naturally decomposed iff it is unitarily equivalent to the tensor product of identifiably single-system Hilbert spaces and their associated algebras.

3.3 Individuation blocks

I will now show that qualitatively individuated systems provide the natural decompositions being sought.

Let us consider what algebra of operators we may associate with a single, qualitatively individuated system (for simplicity I will concentrate on the two-system case). Recall that qualitative individuation is individuation by projectors. So suppose that our two individuation criteria (one for each of the two systems) are given by the projectors $E_\alpha, E_\beta$, each of which acts on the single-system Hilbert space $H$.

It is important that $E_\alpha \perp E_\beta$, i.e. $E_\alpha E_\beta = E_\beta E_\alpha = 0$, so that none of the two systems is individuated by the other’s criterion. (The importance of this condition will soon become clear.) Call the system individuated by $E_\alpha$ the $\alpha$-system, and the system individuated by $E_\beta$, the $\beta$-system.

Now define the following projector on $\mathcal{H}^{(2)}$:

$$\mathcal{E} := E_\alpha \otimes E_\beta + E_\beta \otimes E_\alpha$$ \hspace{1cm} (8)

$\mathcal{E}$ is symmetric, i.e. it satisfies PI, and its range has non-zero components in both the symmetric ($\mathcal{H}_+^{(2)}$) and anti-symmetric ($\mathcal{H}_-^{(2)}$) joint Hilbert spaces.
Figure 1: The intertwiners between the joint Hilbert spaces $\mathcal{E} \left[ \mathcal{H}^{(2)}_+ \right]$, $\mathcal{E} \left[ \mathcal{H}^{(2)}_- \right]$ and $E_\alpha [\mathcal{H}] \otimes E_\beta [\mathcal{H}]$. The square on the left represents $\mathcal{H}^{(2)}_+$ in some product basis and the square on the right represents $\mathcal{H}^{(2)}_- \otimes E_\alpha [\mathcal{H}]$ in the corresponding “symmetry basis”, in which all fermion states (blue) and all boson states (red) are grouped together.

I now claim that both $\mathcal{E} \left[ \mathcal{H}^{(2)}_+ \right]$ (bosonic) and $\mathcal{E} \left[ \mathcal{H}^{(2)}_- \right]$ (fermionic) have natural decompositions into the single-system spaces $E_\alpha [\mathcal{H}]$ and $E_\beta [\mathcal{H}]$. I call these subspaces individuation blocks.

To prove this, we need to establish both that: (i) the single-system spaces have an identifiable single-system interpretation; and (ii) the joint Hilbert spaces $\mathcal{E} \left[ \mathcal{H}^{(2)}_+ \right]$ and $\mathcal{E} \left[ \mathcal{H}^{(2)}_- \right]$ are unitary equivalent to $E_\alpha [\mathcal{H}] \otimes E_\beta [\mathcal{H}]$.

The first condition is satisfied, since $E_\alpha [\mathcal{H}]$ and $E_\beta [\mathcal{H}]$ are just subspaces of the familiar single-system Hilbert space $\mathcal{H}$, and we may preserve all physical interpretations under the projections by the individuation criteria $E_\alpha$ and $E_\beta$.

The second condition is satisfied, since we can provide the explicit forms of the intertwiners between the three joint Hilbert spaces; they are given in Figure 1. (It may be helpful to point out that, while these operators are not unitaries on the whole of $\mathcal{H}$, they are when restricted to their relevant domains so long as $E_\alpha$ and $E_\beta$ are orthogonal.)

In a little more detail, select for example any two single-system operators $A$ and $B$ on $\mathcal{H}$. Then $E_\alpha A E_\alpha \otimes E_\beta B E_\beta$ belongs to the tensor product algebra $B(E_\alpha [\mathcal{H}] \otimes E_\beta [\mathcal{H}]) \cong B(E_\alpha [\mathcal{H}]) \otimes B(E_\beta [\mathcal{H}])$, and is sent to

$$E_\alpha A E_\alpha \otimes E_\beta B E_\beta + E_\alpha A E_\alpha \otimes E_\beta B E_\beta$$

under the intertwiner $U_\pm : E_\alpha [\mathcal{H}] \otimes E_\beta [\mathcal{H}] \rightarrow \mathcal{E} \left[ \mathcal{H}^{(2)}_\pm \right]$, where $U_\pm := \frac{1}{\sqrt{2}} (1 \pm P(12))$. $U_\pm$ also clearly sends product states $|\phi\rangle \otimes |\chi\rangle$, where $|\phi\rangle \in E_\alpha [\mathcal{H}]$, $|\chi\rangle \in E_\beta [\mathcal{H}]$ to states of the form $\frac{1}{\sqrt{2}} (|\phi\rangle \otimes |\chi\rangle \pm |\chi\rangle \otimes |\phi\rangle)$. 

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The foregoing results apply to any subspace $E[\mathcal{H}^{(2)}]$, as defined in (8), so long as $E_\alpha \perp E_\beta$, and we can straightforwardly generalise to assemblies of more than two systems.

A class of instances of qualitative individuation that is of particular interest arises when the single-system Hilbert space $\mathcal{H}$ represents more than one degree of freedom. In this case, if the individuation criteria $E_\alpha, E_\beta$ apply to less than the full degrees of freedom, then the full algebra of linear bounded operators on the remaining degrees of freedom is available to the qualitatively individuated systems.

For simplicity, suppose that $\mathcal{H}$ represents two degrees of freedom; i.e., $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Now let us choose the individuation criteria $E_\alpha = e_\alpha \otimes 1, E_\beta = e_\beta \otimes 1$, where $e_\alpha$ and $e_\beta$ act on $\mathcal{H}_1$ and $1$ is the identity on $\mathcal{H}_2$. From the results above, it follows that the algebra of the $\alpha$-system $\mathcal{A}_\alpha = \mathcal{B}(\text{ran}(E_\alpha)) = \mathcal{B}(\text{ran}(e_\alpha)) \otimes \mathcal{B}(\mathcal{H}_2)$ and the algebra of the $\beta$-system is $\mathcal{A}_\beta = \mathcal{B}(\text{ran}(E_\beta)) = \mathcal{B}(\text{ran}(e_\beta)) \otimes \mathcal{B}(\mathcal{H}_2)$. Thus the full algebra $\mathcal{B}(\mathcal{H}_2)$ is available to both qualitatively individuated systems (cf. Figure 2(b)).

Finally, we may wish to consider individuating by multiple individuation blocks. We may break down the entire joint Hilbert space $\mathcal{H}^{(2)}$ into those subspaces that do count as individuation blocks and those that do not, and then provide natural decompositions for each of the off-diagonal subspaces. Each individuation block is associated with its own pair of qualitatively individuated systems, and thus behaves in its own right like a joint Hilbert space representing an assembly of two distinguishable systems; cf. Figure 2(a).

In more detail: We may decompose the single-system Hilbert space $\mathcal{H}$ using a complete family of projectors $\{E_i\}$, $\sum_i E_i = 1$:

$$\mathcal{H} = \left(\sum_i E_i\right)\mathcal{H} = \bigoplus_i E_i[\mathcal{H}] =: \bigoplus_i \mathcal{H}_i$$

Then, with $S_\pm$ the appropriate symmetry projector, the joint Hilbert space is

$$\mathcal{H}^{(2)}_\pm = S_\pm \left(\bigoplus_i \mathcal{H}_i \otimes 1\right)$$

$$= S_\pm \left(\bigoplus_i \mathcal{H}_i \otimes \bigoplus_i \mathcal{H}_i\right)$$

$$= \bigoplus_i S_\pm (\mathcal{H}_i \otimes \mathcal{H}_i)$$

$$= \bigoplus_i S_\pm (\mathcal{H}_i \otimes \mathcal{H}_i) U_\pm (\mathcal{H}_i \otimes \mathcal{H}_j) U_\pm^\dagger$$

This case corresponds to Huggett and Imbo’s (2009, pp. 315-6) ‘approximately distinguishable’ systems.
Figure 2: (a) The (anti-) symmetric projection of the tensor product of two Hilbert spaces may be decomposed into spaces which exhibit a tensor product structure. (Light grey squares indicate condensed states, which remain under symmetrization but not anti-symmetrization.) (b) If the two Hilbert spaces are decomposed into eigensubspaces of only one degree of freedom, then the “off-diagonal” elements of the decomposition serve as irreps for the full algebra of operators for the other degrees of freedom.

where $U_\pm$ is the intertwiner defined above. Each $(i,j)$ term on the RHS of (14) corresponds to an individuation block.

One might hope to cover, as far as possible, the entire joint Hilbert space $\mathcal{H}_\pm^{(2)}$ with individuation blocks, so that one achieves a decomposition of the full joint space after all. However, these hopes are forlorn, for two reasons. First, in the case of bosons, multiply occupied states (of the form $|\phi\rangle \otimes |\phi\rangle$) will never lie in any individuation block, since it would require individuation criteria to become non-orthogonal. These states are always out of reach. Second, in all cases, the imposition of multiple sets of individuation criteria induces a superselection rule between individuation blocks, since the unitary equivalence results above rely only to each individuation block in isolation. Therefore one cannot impose multiple sets of individuation criteria without losing information about the joint state.

3.4 *Is Permutation Invariance compulsory?*

Here is an appropriate place to make some brief comments about Huggett and Imbo’s (2009, pp. 313) recent claim that it is not necessary to impose permutation invariance on systems with identical “intrinsic” (state-independent) properties. This is because, they claim, systems may be distinguished according to their ‘trajectories’ (i.e. single-system states). If they are correct, this would entail that factorism is, after all, a viable interpretative position for such systems—so long as we understand factor Hilbert space labels as representing these trajectories (just as, in the case of distinguishable systems, we use factor Hilbert space labels to represent distinct state-independent properties of
The results of this section show that Huggett and Imbo are partly correct. In my jargon: they are right that an un-symmetrised Hilbert space is an equally adequate (since unitarily equivalent) means to represent an assembly of qualitatively individuated systems—for those states in which the individuation criteria succeed; and that therefore there is no practical need to impose IP, or, therefore, to repudiate factorism when represented those states and those states alone. But they are wrong to claim that it is not necessary to impose PI to represent all of the available states for systems with identical intrinsic properties. For the unitary equivalence result above, on which Huggett and Imbo’s claim depends, holds only for the appropriate individuation block. As soon as the assembly’s state has components that lie outside of this subspace, the equivalence breaks down.

I must emphasise too that the breakdown of unitary equivalence outside of the relevant individuation block does not just mean that, for states outside this subspace, the two formalisms yield conflicting empirical claims (which come down in favour of PI). Rather, the quasi-factorist formalism ceases to make physical sense for states outside of the relevant individuation block. For, outside of this subspace, the systems no longer occupy the states upon which their individuation—and therefore the entire quasi-factorist formalism—was based.

Huggett and Imbo mistakenly suppose that imposing PI prevents one from qualitatively individuating systems. (As Huggett and Imbo (2009, p. 315) put it: ‘[PI] ⇒ trajectory indistinguishability’.) But that inference assumes what I deny: namely, factorism. Without factorism, we can agree with Huggett and Imbo that systems may be qualitatively individuated, without contravening PI. Moreover: without factorism but with PI, we may represent all of the states available to the assembly, without fear that our representational apparatus will break down.

3.5 Russellian vs. Strawsonian approaches to individuation

All of the results of the previous sections apply only to joint states that have non-zero support within some individuation block. This is the subspace for which the individuation criteria for the systems fully succeeds; i.e. for which the projector

$$E(\alpha, \beta) := E_\alpha \otimes E_\beta + E_\beta \otimes E_\alpha$$

has eigenvalue 1. What about states for which individuation does not succeed? The question is important, since we want a procedure for calculating expectation values of quantities which belong to the joint algebra of the qualitatively individuated quantities; and we want that procedure to be as general as possible.

The way to proceed depends on one’s stance toward reference failure for individuation criteria. I see two equally acceptable routes, which may be associated (a little tenden-tiously) with the classic debate over reference failure for definite descriptions. With a little poetic licence, I call the two routes Russellian and Strawsonian.

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The Russellian route (cf. Russell 1905) takes the claim of success of the individuation criteria \( E_\alpha \) and \( E_\beta \) to be an implicit additional claim to any explicit claim which implements those criteria. Thus the expectation value for any quantity \( Q \in \mathcal{B} \left( E(\alpha, \beta) \left[ \mathcal{F}(2) \right] \right) \) is given by

\[
\langle Q \rangle^{(R)}_{(\alpha, \beta)} := \text{Tr}(E(\alpha, \beta)\rho E(\alpha, \beta)Q),
\]

which uses the usual quantum mechanical specifications for expressing conjunction. But \( Q \) commutes with \( E(\alpha, \beta) \), since it is a sum of products of single-system quantities, each of which commutes with \( E_\alpha \) and \( E_\beta \). So (16) may be simplified to \( \text{Tr}(\rho E(\alpha, \beta)Q) \).

The Strawsonian route (cf. Strawson 1950) instead takes the joint success of the individuation criteria \( E_\alpha \) and \( E_\beta \) as a presupposition of any claim which uses that strategy. Therefore any expectation values calculated under the presupposition of the success of \( E(\alpha, \beta) \) must be renormalized by conditionalizing on that success. This is done using the usual Lüder rule

\[
\rho \mapsto \rho_{\alpha\beta} := \frac{E(\alpha, \beta)\rho E(\alpha, \beta)}{\text{Tr}(\rho E(\alpha, \beta))}.
\]

The expectation value of any quantity \( Q \in \mathcal{B} \left( E(\alpha, \beta) \left[ \mathcal{F}(2) \right] \right) \) is then given simply by

\[
\langle Q \rangle^{(S)}_{(\alpha, \beta)} := \text{Tr}(\rho_{\alpha\beta}Q).
\]

Note that conditionalization requires that \( \text{Tr}(\rho E(\alpha, \beta)) > 0 \), which means that the state must have some support in the individuation block. The fact that \( \text{Tr}(\rho_{\alpha\beta}Q) \) is undefined when \( \text{Tr}(\rho E(\alpha, \beta)) = 0 \) meshes rather nicely with Strawson’s well-known claim that statements containing failed definite descriptions do not possess a truth value.

It will have been noted that the difference between the Russellian and Strawsonian routes for expectation values lies only in the multiplicative factor \( \frac{1}{\text{Tr}(\rho E(\alpha, \beta))} \). A point in favour of the Strawsonian approach is that the identity \( \mathbf{1}_{\mathcal{F}(2)} \) has expectation value \( \langle \mathbf{1}_{\mathcal{F}(2)} \rangle_{(\alpha, \beta)}^{(S)} = 1 \) for all normalizable states, while under the Russellian route the identity’s expectation value is equal to \( E(\alpha, \beta) \)’s expectation value, which may take any value from 0 to 1. A point in favour of the Russellian approach is that expectation values may be defined for all states. On this approach, if the assembly’s state has no support in the individuation block, then expectation value for every quantity is zero.

3.6 *Qualitative individuation over time

In this section I take a brief look at individuation criteria which are using over a period of time. The investigation here will be all too brief, but will hopefully give a flavour of the direction of future investigation.

Let \( \mathcal{E} := E_\alpha \otimes E_\beta + E_\beta \otimes E_\alpha \) represent our individuation criteria for two particles at time \( t = 0 \). The obvious way to turn this into evolving individuation criteria is to use the usual Heisenberg prescription for time-dependent quantities. Thus, if \( U(t) = e^{-it\mathcal{H}} \),
is the evolution operator for the assembly, with Hamiltonian $H$, then we may define the
time-dependent version of the projector $\mathcal{E}$ (defined in (8), above):

$$\mathcal{E}(t) = U(t)\mathcal{E}U^\dagger(t)$$

(18)

The expectation value of $\mathcal{E}(t)$ is a constant of the motion:

$$\langle\mathcal{E}(t)\rangle_t = \text{Tr}(U(t)\rho U^\dagger(t)U(t)\mathcal{E}U^\dagger(t)) = \text{Tr}(\rho\mathcal{E}) = \langle\mathcal{E}\rangle_0,$$

(19)

so if the individuation criteria succeed at $t = 0$, then the dynamics ensure that they
succeed for all times. In other words, any state that lies in an individuation block stays
there under dynamical evolution.

This proposal may be seen as analogous to classical mechanical individuation pro-
cedures for point particles in reduced phase space (see Belot (2001)). Working in the
reduced phase space, by evolving any equivalence class of system points along the Hamil-
tonian flow we achieve natural trans-temporal identifications for the point particles by
demanding continuous trajectories for each particle.

However, unlike in the classical case, we cannot guarantee that the time-evolute of
$\mathcal{E}$ has the right features to count as a pair of individuation criteria for the two particles.
For this it is necessary and sufficient that

$$U(t)\left( E_\alpha \otimes E_\beta + E_\beta \otimes E_\alpha \right) U^\dagger(t) = E_\alpha(t) \otimes E_\beta(t) + E_\beta(t) \otimes E_\alpha(t),$$

(20)

where, for all times $t$, $E_\alpha(t) \perp E_\beta(t)$. That is, it is necessary and sufficient that $\mathcal{E}$’s evolu-
tion may be expressed in terms of a piecemeal evolution of the single-system projectors
$E_\alpha$ and $E_\beta$.

To see that condition (20) does not hold generally, one need only consider an evolution
that takes a heterogeneous state of two bosons at $t = t_i$ to a product state with identical
factors at $t = t_f$. In this case, at time $t_i$, $E_\alpha(t_i) \perp E_\beta(t_i)$; but by $t = t_f$, we have $E_\alpha(t_f) = E_\beta(t_f)$, which contradicts the requirement that individuation criteria for distinct systems
be orthogonal.

I know of no general result which gives conditions on either the evolution $U(t)$ or the
assembly’s state, for the satisfaction of (20). However, it is easily seen that the following
two special cases entail it:

1. the dynamical evolution commutes with the individuation criteria, i.e. $[U(t), \mathcal{E}] = 0$;
2. the dynamical evolution is factorizable, i.e. $U(t) = W(t) \otimes W(t)$.

I will now discuss these cases in a little more detail.

*The dynamical evolution commutes with the individuation criteria.* I.e. $[U(t), \mathcal{E}] = 0$,
in which case $\mathcal{E}(t) = \mathcal{E}(0)$, for all times $t$, so (20) is satisfied because the RHS is just
This case is in some sense more interesting than case 2, since it permits interactions between the constituent systems (whereas factorizable evolutions do not).

Consider for example two electrons where $U(t)$ describes evolution due to mutual, purely electrostatic repulsion—i.e. it acts non-trivially only on spatial degrees of freedom and there is no spin-orbit coupling—while the individuation criteria pick out non-spatial single-system states, e.g. spin in some direction. Thus if $U(t) = V(t)_{\text{space}} \otimes 1_{\text{spin}}$ (where $V(0) = 1_{\text{space}}$), then the individuation criteria

$$E_\alpha := 1_{\text{space}} \otimes |1\rangle\langle 1|_{\text{spin}}; \quad E_\beta := 1_{\text{space}} \otimes |\downarrow\rangle\langle \downarrow|_{\text{spin}}$$

continue to succeed in qualitatively individuating the two electrons: that is, for any joint state $|\Psi\rangle$, $\langle \Psi(t)|E|\Psi(t)\rangle = \langle \Psi|U^\dagger(t)E U(t)|\Psi\rangle = \langle \Psi|E|\Psi\rangle$. It ought to be emphasized that this holds even when and after the spatial part of the joint state evolves under $V(t)$ to a separable state with identical factors (in which case the spin part of the joint state will be the singlet state), so that for some time the electrons’ spatial wave-functions totally overlap.

This case is also considered by Ladyman et al (2013, 217-8) in the special case in which, for some initial time $t_i$, the joint state is

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}} (|L, \uparrow\rangle \otimes |R, \downarrow\rangle - |R, \downarrow\rangle \otimes |L, \uparrow\rangle)$$

(22)

where $|L, \uparrow\rangle := |L\rangle \otimes |\uparrow\rangle$, etc. and $|L\rangle$ is a single-electron spatial state concentrated in a region $L$, etc. (It may be checked that $\langle \Psi_i|E|\Psi_i\rangle = 1$ for individuation criteria as chosen in (21).) If we let this state evolve under $U(t_f - t_i)$ and measure its amplitude for the “outward” state

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}} (|L', \uparrow\rangle \otimes |R', \downarrow\rangle - |R', \downarrow\rangle \otimes |L', \uparrow\rangle)$$

(23)

(note that $\langle \Psi_f|E|\Psi_f\rangle = 1$), then we obtain

$$\langle |\Psi_f\rangle|U(t_f - t_i)|\Psi_i\rangle = \langle L'| \otimes (R'| V(t_f - t_i)|L\rangle \otimes |R\rangle.$$ (24)

Ladyman et al (2013, 218) note that this is the same amplitude as would be obtained for two “distinguishable” systems evolving under the same time evolution, but for which the “inward” and “outward” states are

$$|\Psi_i'\rangle = |L, \uparrow\rangle \otimes |R, \downarrow\rangle; \quad |\Psi_f'\rangle = |L', \uparrow\rangle \otimes |R', \downarrow\rangle.$$ (25)

Given the results of section 3 and the unitary equivalence between the “indistinguishable” and “distinguishable” cases on which they are based, this is to be expected: the intertwiner defined by $E_\alpha$ and $E_\beta$ in (21) connects the states $|\Psi_i\rangle$ and $|\Psi_i'\rangle$, and $|\Psi_f\rangle$ and $|\Psi_f'\rangle$, respectively. And I have advocated giving the same physical interpretation in both cases.

However, Ladyman et al (2013, 218) draw a different conclusion:
Note that in this case [in which spin degrees of freedom are considered,] the interference term disappears [making \( \langle \Psi_f | U(t_f - t_i) | \Psi_i \rangle = \langle \Psi'_f | U(t_f - t_i) | \Psi'_i \rangle \), producing agreement between the “indistinguishable” and “distinguishable” cases]. The lack of interference is usually explained by the fact that it is in principle possible to tell which of the two electrons was registered at the region \( L' \) (\( R' \)) by measuring its spin. However, this explanation seems to be not entirely accurate. Due to the antisymmetrization the electrons are never properly identifiable. The interference disappears because we can correlate the property of being located in the region \( L \) at the moment \( t_i \) with the property of being located in \( L' \) at \( t_f \), and not because at \( t_i \) the region \( L \) is occupied by a particle which is numerically distinct from any other particle in the universe, and which after the collision is located in \( L' \).

It will hopefully be clear by now where Ladyman et al go wrong here: the very thing that correlates the property of being located in \( L \) at \( t_i \) with the property of being located in \( L' \) at \( t_f \)—that is, the property of being spin-up—is precisely what makes it ‘in principle possible to tell which of the two electrons was registered at the region \( L' \) by measuring its spin’: it is the spin-up electron. Ladyman et al, while motivated by anti-factorist concerns (see particularly their section 2), succumb at the crucial juncture to a factorist interpretation, which under anti-symmetrization forbids identifying electrons with their states, since they must already be identified by the factor Hilbert spaces. This forces them to say wrongly that ‘the electrons are never properly identifiable’ and vetoes the most natural interpretation of their results.

The dynamical evolution is factorizable. In this case we have \( U(t) = W(t) \otimes W(t) \), where \( W(t) \) is a continuous one-parameter family of unitaries on the single-system Hilbert space \( \mathcal{H} \). Under this evolution, the two constituent systems cannot properly be said to interact, and the purity of single-system states is preserved over time.

Factorizable evolutions present a problem: there may be no unique pair of time-dependent individuation criteria \( E_\alpha(t), E_\beta(t) \) for which condition (20) is satisfied. This result is as bad as there being no such pair, if our goal is to find uniquely natural trans-temporal identity conditions.

The fact that uniqueness is not guaranteed for factorizable evolutions is illustrated by the following simple example. Consider the singlet state for two spin-\( 1/2 \) fermions:

\[
|\psi_-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle),
\]

and the evolution \( U(t) = e^{-iE t} \), where \( E \) is the Hamiltonian eigenvalue associated with \( |\psi_-\rangle \) (given PI, \( |\psi_-\rangle \) must be a stationary state). And suppose that at time \( t = 0 \) we individuate two systems using the projectors \( |\uparrow\rangle\langle\uparrow| \) and \( |\downarrow\rangle\langle\downarrow| \). We now seek time-dependent individuation criteria for these two systems. Condition (20) suggests the time-dependent projectors \( W(t)|\uparrow\rangle\langle\uparrow| W(t)^\dagger \) and \( W(t)|\downarrow\rangle\langle\downarrow| W(t)^\dagger \), where \( U(t) = W(t) \otimes W(t) \) for some unitary \( W(t) \) on \( \mathcal{H} \). But \( W(t) \) is under-determined by this requirement.
We make use of the group isomorphism $U(2) \otimes U(2) \cong U(3) \oplus U(1)$. Let $w(t)$ be any continuous one-parameter family of unitary $2 \times 2$ matrices

$$w(t) = \begin{pmatrix} \alpha(t)e^{i\phi(t)} & \beta(t)e^{i\phi(t)} \\ -\beta^*(t) & \alpha^*(t) \end{pmatrix} \tag{27}$$

where $|\alpha(t)|^2 + |\beta(t)|^2$ for all $t$. Then

$$w(t) \otimes w(t) = \begin{pmatrix} \alpha^2(t)e^{2i\phi(t)} & \alpha(t)\beta(t)e^{2i\phi(t)} & \alpha(t)\beta(t)e^{2i\phi(t)} & \beta^2(t)e^{2i\phi(t)} \\ -\alpha(t)\beta^*(t)e^{i\phi(t)} & |\alpha(t)|^2e^{i\phi(t)} & -|\beta(t)|^2e^{i\phi(t)} & \alpha^*(t)\beta(t)e^{i\phi(t)} \\ -\alpha(t)\beta^*(t)e^{i\phi(t)} & -|\beta(t)|^2e^{i\phi(t)} & |\alpha(t)|^2e^{i\phi(t)} & \alpha^*(t)\beta(t)e^{i\phi(t)} \\ [\beta^*(t)]^2 & -\alpha^*(t)\beta^*(t) & -\alpha^*(t)\beta^*(t) & [\alpha^*(t)]^2 \end{pmatrix}$$

(28)

With a suitable change of basis this becomes

$$w(t) \otimes w(t) = \begin{pmatrix} \alpha^2(t)e^{2i\phi(t)} & \alpha(t)\beta(t)e^{2i\phi(t)} & \alpha(t)\beta(t)e^{2i\phi(t)} & \beta^2(t)e^{2i\phi(t)} \\ -\alpha(t)\beta^*(t)e^{i\phi(t)} & |\alpha(t)|^2e^{i\phi(t)} & -|\beta(t)|^2e^{i\phi(t)} & \alpha^*(t)\beta(t)e^{i\phi(t)} \\ -\alpha(t)\beta^*(t)e^{i\phi(t)} & -|\beta(t)|^2e^{i\phi(t)} & |\alpha(t)|^2e^{i\phi(t)} & \alpha^*(t)\beta(t)e^{i\phi(t)} \\ [\beta^*(t)]^2 & -\alpha^*(t)\beta^*(t) & -\alpha^*(t)\beta^*(t) & [\alpha^*(t)]^2 \end{pmatrix}$$

(29)

The new basis is such that $w(t) \otimes w(t)$ is decomposed into symmetric components (the $3 \times 3$ matrix at top-left) and anti-symmetric components (the $c$-number at bottom-right).

Therefore, the restriction of $w(t) \otimes w(t)$ to the anti-symmetric sector $\mathcal{A}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, spanned by $|\psi_\rightarrow\rangle(t)$, is $e^{i\phi(t)}$. So if we make the identification $W(t) = w(t)$, we have $U(t) = e^{i\phi(t)}$ on $\mathcal{A}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Thus $W(t)$ may be any unitary $2 \times 2$ matrix of the form given in Equation (27), subject only to the requirement that $\phi(t) = -Et$. So $W(t)e^{iEt}$ may be any smooth $SU(2)$-valued function of time. The geometrical interpretation of this is that the relevant spin axis along which the two systems are individuated (e.g. spin-up vs. spin-down, spin-left vs. spin-right, etc.) may sweep around space however we care to choose (so long as it does so smoothly).

The upshot is that the trans-temporal identity conditions for constituent systems in the trajectory $|\psi_\rightarrow\rangle(t)$ are subject to somewhat the same problems as beset trans-temporal identification of the parts of a spinning sphere of uniform, continuous matter (cf. e.g. Zimmerman (1998), Butterfield (2006)). In both cases no uniquely correct or natural trans-temporal identity conditions (which, in the case of the spinning sphere, would determine its angular velocity) are available. (In section 7.2, this analogy is strengthened by the identification of the joint state $|\psi_\rightarrow\rangle$ with the Bloch sphere defined by the single-system states $|\uparrow\rangle$ and $|\downarrow\rangle$.)

To sum up: Evolutions which commute with a family of individuation criteria chosen for some time allow individuation over time, since in this case successful individuation at one time entails successful individuation at all times. Factorizable evolutions give favourable conditions under which trans-temporal individuation criteria for constituent systems may be defined; but the conditions are “too favourable”: the criteria may not
be unique. We may even combine the two cases, and the two examples discussed above. Thus if we decide to individuate the two electrons above according to their spin along the $z$-axis, then we have one way of keeping track of them over time; but we may decide a different spin axis—even one that changes over time—which allows us equally successfully to individuate two electrons over time. Each resulting history, told as the evolving states of two electrons, is as correct as any other. This is a puzzling result, and will be addressed in section 7. But first, I will continue my exposition of qualitative individuation.

4 Individuating single systems

In this section, I turn away from the problem of completely decomposing an assembly into constituent systems, and turn instead to the problem of picking out a single constituent system from the assembly. I seek a means to calculate expectation values for quantities associated with a single qualitatively individuated system, whose individuation criterion we may choose.

4.1 Expectation values for constituent systems

The way I will proceed is inspired in part by the Strawsonian approach to individuation in Section 3.5. The main idea, there and here, is to conditionalize upon the success of the individuation. As usual, I work, for the sake of simplicity, in the $N = 2$ case (unless otherwise stated); the generalization to $N > 2$ will be obvious.

We begin with a chosen individuation criterion, a projector $E_\alpha$, which acts on the single-system Hilbert space $\mathcal{H}$. Then it may be checked that the operator

$$n_\alpha := E_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes E_\alpha$$

is a number operator for the two-system assembly’s Hilbert space. That is, it “counts” the number of systems which are picked out by $E_\alpha$. It is represented in Figure 3.

We now define a linear map $\pi_\alpha : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(S_\pm^{(2)})$ from the single-system algebra into the joint algebra of the assembly, which takes single-system operators and gives the appropriate operator on the joint Hilbert space associated with the $\alpha$-system:

$$\pi_\alpha(Q) := E_\alpha Q E_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes E_\alpha Q E_\alpha.$$  \hfill (31)

(Note that, if $Q = E_\alpha Q E_\alpha$, then $\pi_\alpha$ is just the symmetrizer for $Q$.)

I now claim that the expectation value for any single-system quantity $Q$, associated with “the” $\alpha$-system is given by

$$\langle Q \rangle_\alpha := \frac{\langle \pi_\alpha(Q) \rangle}{\langle n_\alpha \rangle}.$$  \hfill (32)

I will establish this claim by considering a few examples.
The state of the assembly $|\psi\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle \pm |\beta\rangle \otimes |\alpha\rangle)$, where $E_\alpha|\alpha\rangle = |\alpha\rangle$ and $E_\alpha|\beta\rangle = 0$, and $Q|\alpha\rangle = q|\alpha\rangle$. Then $\langle n_\alpha \rangle = 1$ and $\langle \pi_\alpha(Q) \rangle = q$; so $\langle Q \rangle_\alpha = q$.

1. The state of the assembly $|\psi\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle \pm |\beta\rangle \otimes |\alpha\rangle)$, where $E_\alpha|\alpha\rangle = |\alpha\rangle$ and $E_\alpha|\beta\rangle = 0$, and $Q|\alpha\rangle = q|\alpha\rangle$. Then $\langle n_\alpha \rangle = 1$ and $\langle \pi_\alpha(Q) \rangle = q$; so $\langle Q \rangle_\alpha = q$.

That is, the system individuated by $E_\alpha$ takes as its expectation for $Q$ the value $q$, associated with the state $|\alpha\rangle$, for which individuation succeeds (i.e., the state that it is in the range of $E_\alpha$). (Indeed, the $\alpha$-system is in an eigenstate for $Q$, since $\langle Q^2 \rangle_\alpha = q^2$.)

2. $|\psi\rangle = c_1 \frac{1}{\sqrt{2}} (|\alpha_1\rangle \otimes |\beta_1\rangle \pm |\beta_1\rangle \otimes |\alpha_1\rangle) + c_2 \frac{1}{\sqrt{2}} (|\alpha_2\rangle \otimes |\beta_2\rangle \pm |\beta_2\rangle \otimes |\alpha_2\rangle)$, where $|c_1|^2 + |c_2|^2 = 1$; and for all $i = 1, 2$: $E_\alpha|\alpha_i\rangle = |\alpha_i\rangle$ and $E_\alpha|\beta_i\rangle = 0$, and $Q|\alpha_i\rangle = q_i|\alpha_i\rangle$.

Then $\langle n_\alpha \rangle = 1$ and $\langle \pi_\alpha(Q) \rangle = |c_1|^2 q_1 + |c_2|^2 q_2$; so $\langle Q \rangle_\alpha = |c_1|^2 q_1 + |c_2|^2 q_2$. That is, the system individuated by $E_\alpha$ takes as its expectation for $Q$ the average for all single-system states $|\alpha_i\rangle$, for which the individuation succeeds. The weights for this average are given by the relative amplitudes of the non-GM-entangled terms.

3. $|\psi\rangle = |\alpha\rangle \otimes |\alpha\rangle$, for $|\alpha\rangle$ as above. Then $\langle n_\alpha \rangle = 2$ and $\langle \pi_\alpha(Q) \rangle = 2q$; so $\langle Q \rangle_\alpha = q$.

In this case, $E_\alpha$ individuates two systems, and the expectation (indeed, eigenvalue) for $Q$ for both of them is $q$.

4. $|\psi\rangle = \frac{1}{\sqrt{2}} (|\alpha_1\rangle \otimes |\alpha_2\rangle \pm |\alpha_2\rangle \otimes |\alpha_1\rangle)$, for $|\alpha_1\rangle, |\alpha_2\rangle$ as above. Then $\langle n_\alpha \rangle = 2$ and $\langle \pi_\alpha(Q) \rangle = q_1 + q_2$; so $\langle Q \rangle_\alpha = \frac{1}{2}(q_1 + q_2)$. In this case, $E_\alpha$ again individuates two systems, one whose expectation value for $Q$ is $q_1$, and one whose value is $q_2$; thus we take the average. However, the weights for this average are not given, as above, by relative amplitudes for non-GM-entangled terms; (the entire state is non-GM-entangled). Rather, they are given by the relative frequency, in a single non-GM-entangled state, of each single-system state for which individuation succeeds.

5. $|\psi\rangle = c_1 U_+ (|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes |\beta_1\rangle) + c_2 U_- (|\alpha_3\rangle \otimes |\beta_1\rangle \otimes |\beta_2\rangle)$, where $U_\pm$ is the in-
tortuous objection to the way given by (32). However, I submit, there are no clear intuitions to rely on in this case, and I can see no way to calculate statistical weights from the relative amplitudes and relative frequencies. Least for cases 1 to 4. Case 5 seems to me less clear cut, since one might favour a different system qualitatively individuated by $E$.

Thus my claim—that the expectation value of any single-system quantity $Q$, for the system qualitatively individuated by $E_\alpha$, is given by (32)—yields the right results, at least for cases 1 to 4. Case 5 seems to me less clear cut, since one might favours a different system qualitatively individuated by $E$. Thus the map $\pi_\alpha$ does not constitute an isomorphism between the single-system algebra $\mathcal{B}(\mathcal{H})$ or indeed any subalgebra thereof and the range of $\pi_\alpha$. It is not even a homomorphism. For example, it may be checked that $\pi_\alpha(AB) \neq \pi_\alpha(A)\pi_\alpha(B)$ does not hold, even for all those $A, B \in \mathcal{B}(\mathcal{H})$ that commute with $E_\alpha$.

This is not an objection to (32), and should come as no surprise. For there are states of the assembly in which $E_\alpha$ fails to individuate a unique system (cf. examples 3-5, above). In these states, we should not expect that $\pi_\alpha(AB) = \pi_\alpha(A)\pi_\alpha(B)$. To perform the operation $B$, followed by $A$, on a given $\alpha$-system (corresponding to $\pi_\alpha(AB)$) relies on a re-identification of that system (and that system alone) after we have operated with $B$. But the individuation criterion $E_\alpha$ cannot be guaranteed to pick out that very same system, if more than one system is picked out by $E_\alpha$.

On the other hand, it may be checked that, for any two states $|\psi\rangle$ of the assembly that are eigenstates of $n_\alpha$ with eigenvalue 1—i.e., for all states in which exactly one system is individuated by $E_\alpha$—we have $\langle\psi|\pi_\alpha(AB)|\psi\rangle = \langle\psi|\pi_\alpha(A)\pi_\alpha(B)|\psi\rangle$, as expected.

### 4.2 Reduced density operators

Thus we have a recipe for calculating the expectation value of any single-system quantity for a qualitatively individuated system or systems. It remains for me to give a general prescription for calculating the reduced density operator for such a system. We require that the reduced density operator $\rho_\alpha$ satisfy the condition that, for all $Q \in \mathcal{B}(\mathcal{H})$: $\text{tr}(\rho_\alpha Q) = \langle Q \rangle_\alpha$, as given in (32) (I use the expression ‘tr’ with a lowercase ‘t’ to denote the trace over the single-system Hilbert space $\mathcal{H}$). We know from Gleason’s Theorem that a unique such operator exists.

As usual, it is helpful to work by analogy with the case of “distinguishable” systems. The usual prescription for the reduced density operator of a constituent system, say the $\pi_\alpha$...
kth, of the assembly is (with \( \rho \) the state of the assembly):

\[
\rho_k := \text{Tr}_k (\rho),
\]

where \( \text{Tr}_k \) denotes a partial trace over all but the \( k \)th factor Hilbert space. Now this prescription is obviously no use to anti-factorists; but an equivalent formulation to (33) exists that will be of far more use. First we choose a complete orthobasis \( \{|\phi_i\rangle\} \) for the single-system Hilbert space \( \mathcal{H} \). Then

\[
\rho_k := \sum_{i,j} \text{Tr} \left( \rho |\phi_j\rangle\langle \phi_i| \right) |\phi_i\rangle\langle \phi_j|
\]

where \( d := \dim(\mathcal{H}) \) and

\[
|\phi_j\rangle\langle \phi_i|_k := \bigotimes_{1}^{k-1} 1 \otimes |\phi_j\rangle \otimes \bigotimes_{N-k}^{N} 1
\]

and we now perform a full trace on the joint Hilbert space in (34).

We may adapt (34) for qualitatively individuated systems in the following way. First, we replace each operator \( |\phi_j\rangle\langle \phi_i|_k \), which is indexed to a factor Hilbert space, with \( \pi_\alpha(|\phi_j\rangle\langle \phi_i|) \), as given in (31). And second, we “conditionalize” by dividing by \( \langle n_\alpha \rangle = \text{Tr}(\rho n_\alpha) \). Thus

\[
\rho_\alpha = \frac{1}{\langle n_\alpha \rangle} \sum_{i,j} \text{Tr} \left[ \rho \pi_\alpha(|\phi_j\rangle\langle \phi_i|) \right] |\phi_i\rangle\langle \phi_j|
\]

Written out in full, and for any \( N \), we have

\[
\rho_\alpha = \frac{1}{\langle n_\alpha \rangle} \sum_{i,j} \text{Tr} \left[ \rho \left( \sum_{k=1}^{N} \bigotimes_{1}^{k-1} 1 \otimes E_\alpha |\phi_j\rangle \otimes E_\alpha \otimes \bigotimes_{N-k}^{N} 1 \right) \right] \text{Tr} \left[ \rho \left( \sum_{k=1}^{N} \bigotimes_{1}^{k-1} 1 \otimes E_\alpha \otimes \bigotimes_{N-k}^{N} 1 \right) \right]
\]

We may arrange for the first \( d_\alpha := \dim(E_\alpha) \) basis states to span \( E_\alpha[\mathcal{H}] \); in which case

\[
\rho_\alpha = \frac{1}{\langle n_\alpha \rangle} \sum_{i,j} \text{Tr} \left[ \rho \left( \sum_{k=1}^{N} \bigotimes_{1}^{k-1} 1 \otimes |\phi_j\rangle \otimes \bigotimes_{N-k}^{N} 1 \right) \right] \text{Tr} \left[ \rho \left( \sum_{k=1}^{N} \bigotimes_{1}^{k-1} 1 \otimes |\phi_i\rangle \otimes \bigotimes_{N-k}^{N} 1 \right) \right]
\]

We may then be shown (as required) that for any \( Q \in \mathcal{B}(\mathcal{H}) \), \( \text{tr}(\rho_\alpha Q) = \langle Q \rangle_\alpha \). For
this, let \( \{ |\xi_i\rangle \} \) be a complete eigenbasis for \( Q \), where \( Q|\xi_i\rangle = q_i|\xi_i\rangle \). Then

\[
\text{tr}(\rho_\alpha Q) = \frac{1}{\langle n_\alpha \rangle} \sum_{i,j,k}^d \text{Tr}[\rho \pi_\alpha (|\xi_j\rangle\langle\xi_i|)] \langle\xi_k|Q|\xi_i\rangle
\]  

(39)

\[
= \frac{1}{\langle n_\alpha \rangle} \sum_{i,j,k}^d q_k \text{Tr}[\rho \pi_\alpha (|\xi_j\rangle\langle\xi_i|)] \delta_{ki} \delta_{jk}
\]  

(40)

\[
= \frac{1}{\langle n_\alpha \rangle} \sum_k^d \text{Tr}[\rho q_k \pi_\alpha (|\xi_k\rangle\langle\xi_k|)]
\]  

(41)

\[
= \frac{1}{\langle n_\alpha \rangle} \text{Tr}(\rho \pi_\alpha (Q))
\]  

(42)

\[
= 1 \langle n_\alpha \rangle \text{Tr}(\rho \pi_\alpha (Q))
\]  

(43)

\[
=: \langle Q \rangle_\alpha.
\]  

(44)

Remember that \( \rho_\alpha \) as given in (36) and (37) is the average state of any system individuated by \( E_\alpha \). So long as the state \( \rho \) is an eigenstate of \( n_\alpha \) with eigenvalue 1—or even a superposition of \( n_\alpha = 0 \) and \( n_\alpha = 1 \) eigenstates—then \( \rho_\alpha \) yields the state of the \( \alpha \)-system. However, if \( \rho \) contains eigenstates with \( n_\alpha > 1 \), then the interpretation of \( \rho_\alpha \) as the state of the \( \alpha \)-system can no longer be sustained, since in those terms we effectively average over all systems picked out by \( E_\alpha \).

To conclude this section, I note two important limiting cases of Equation (36). The first is when we are maximally discriminating in our individuation; i.e., where \( E_\alpha \) is a one-dimensional projector. Let \( |\alpha\rangle \) be the state for which \( E_\alpha |\alpha\rangle = |\alpha\rangle \). Then, so long as \( \langle n_\alpha \rangle > 0 \), \( \rho_\alpha = |\alpha\rangle\langle\alpha| \), which is to be expected. In this case the \( \alpha \)-system is a Fock space quantum, the object that is added to the field with the creation operator \( a^\dagger(\alpha) \).

The second limiting case lies at the other extreme, in which we individuate with maximum indiscriminateness, i.e. with \( E_\alpha = 1 \). In this case \( \langle n_\alpha \rangle = N \) and \( \pi_\alpha(A) = \sum_{k=1}^N \otimes^{k-1} 1 \otimes A \otimes \otimes^{N-k} 1 \); so

\[
\rho_\alpha = \frac{1}{N} \sum_{i,j}^d |\phi_i\rangle\langle\phi_j| \text{Tr} \left[ \rho \left( \sum_{k=1}^N k^{-1} \otimes |\phi_j\rangle\langle\phi_i| \otimes \otimes^{N-k} 1 \right) \right]
\]  

(45)

\[
= \frac{1}{N} \sum_{k=1}^N \sum_{i,j}^d |\phi_i\rangle\langle\phi_j| \text{Tr} \left[ \rho \left( k^{-1} \otimes |\phi_j\rangle\langle\phi_i| \otimes \otimes^{N-k} 1 \right) \right]
\]  

(46)

\[
= \frac{1}{N} \sum_{k=1}^N \sum_{i,j}^d |\phi_i\rangle\langle\phi_j| \text{Tr} \left[ \rho (|\phi_j\rangle\langle\phi_i|_k) \right] \quad \text{(from (35))}
\]  

(47)

\[
= \frac{1}{N} \sum_{k=1}^N \rho_k \quad \text{(from (34))}.
\]  

(48)
So with maximum indiscriminateness \( \rho_\alpha \) is the “average” of the standard reduced density operators obtained by partial tracing. However, under PI we of course have \( \rho_1 = \rho_2 = \ldots = \rho_k =: \overline{\rho} \), in which case \( \rho_\alpha = \overline{\rho} \). Thus we may say that, under PI, standard reduced density operators obtained by partial tracing codify only the state of the average system, and not the state of any particular system. In this sense, the factorist may be accused of committing the fallacy of misplaced concreteness, as when one takes the “average taxpayer” to be a real person.

The definition of \( \overline{\rho} \) as the average reduced state obtained by partial trace also allows a more elegant expression for the reduced density operator \( \rho_\alpha \) for the \( \alpha \)-system. From (38), we have

\[
\rho_\alpha = \sum_{i,j} E_\alpha |\phi_i\rangle \langle \phi_j| \sum_{k=1}^{N-1} \text{Tr} \left[ \rho \left( \prod_{k=1}^{N-1} 1 \otimes |\phi_j\rangle \langle \phi_i| \otimes \prod_{N-k}^{N-1} 1 \right) \right]
\]

(49)

\[
= \sum_{i,j} E_\alpha |\phi_i\rangle \langle \phi_j| \sum_{k=1}^{N-1} \text{Tr} \left[ \rho \left( \prod_{k=1}^{N-1} 1 \otimes |\phi_j\rangle \langle \phi_i| \otimes \prod_{N-k}^{N-1} 1 \right) \right]
\]

(50)

\[
= \frac{E_\alpha \overline{\rho} E_\alpha}{\text{tr}(\overline{\rho} E_\alpha)}.
\]

(51)

Thus \( \rho_\alpha \) is obtained from the “average state” \( \overline{\rho} \) by the usual L"uder’s rule, where we conditionize on the single-system’s state lying in \( E_\alpha[\mathcal{H}] \).

### 4.3 Qualitative individuation in tensor notation

Some of the fairly ugly notation used above has a more elegant form in abstract index tensor notation (as used e.g. by Geroch (2005)). In this notation, a single-system state is written as a rank-(0,1) tensor, \( \psi^a := |\psi\rangle \) and its dual as a rank-(1,0) tensor, \( \psi_a := \langle \psi| \).

Thus the inner product is given by \( \langle \phi|\psi\rangle = \phi^a \psi_a \) (where summation is assumed for repeated indices; note that \( \phi^a \psi_a = (\phi^a \psi_a)^* \)).

Single-system operators are rank-(1,1) tensors \( Q = Q^b_a \) and matrix elements by \( \langle \phi|Q|\psi\rangle = \phi^m Q^m_n \psi_n \). If the single-system state is given by a density operator \( \rho^b_a \), then the expectation value of \( Q^b_a \) is given by \( \text{tr}(\rho^b_a Q^b_a) = \rho^m_a Q^m_n \).

\( N \)-system states \( \psi^{a_1 a_2 \ldots a_n} \) are rank-(\( N,0 \)) tensors. Supposing the 2-system state \( \psi^{ab} = \sum_{i,j} c_{ij} \psi^a_i \psi^b_j \), the contraction \( \phi_n \psi^{na} := \sum_j (\sum_i c_{ij} \phi_n \psi^i_j) \psi^a_j = \sum_j (\sum_i c_{ij} \langle \phi|\psi_i\rangle \psi^a_j) \), and so on.
Symmetric and anti-symmetric combinations of tensor products are then written as 
\[ \phi[a|\psi|b]_{\pm} := \frac{1}{2} (\phi[a|\psi|b] \pm \phi[a|\psi|b]) \text{, etc.} \] (Note that if \( \phi \) and \( \psi \) are normalized, then their normalized joint state is \( \sqrt{\frac{2}{1+|\langle\phi|\psi\rangle|}} \phi[a|\psi|b]_{\pm} \)). All N-system states \( \psi[a_{1},...,a_{N}] \in S(N) \) satisfy \( \psi[a_{1},...,a_{N}] = \psi[a_{1}|a_{2}|...|a_{N}] \pm \).

We also have to define the N-system symmetrizer \( \Sigma(N) \), which acts on an arbitrary rank-(N, N) tensor \( Q[a_{1}a_{2}|...|a_{N}] \) to produce

\[ \Sigma(N) \left( Q[a_{1}a_{2}|...|a_{N}] \right) := \frac{1}{N!} \sum_{\pi \in S(N)} \left( P(\pi)QP^{\dagger}(\pi) \right)_{a_{1}a_{2}|...|a_{N}} = \frac{1}{N!} \sum_{\pi \in S(N)} Q[a_{\pi(1)}a_{\pi(2)}|...|a_{\pi(N)}] \]  

Note that \( \Sigma(N) \) correlates permutations between upstairs and downstairs indices, so \( \Sigma(N) \left( Q[a_{1}a_{2}|...|a_{N}] \right) \neq Q[a_{1}a_{2}|...|a_{N}] \).

Then, for “distinguishable” systems, given any N-system state \( \psi[a_{1}a_{2}|...|a_{N}] \in S(N) \), the reduced density operator associated system \( k \), obtained by a partial trace, is given by

\[ \rho(k)_{a} := \psi_{n_{1}|...|n_{k-1}|a} n_{k+1|...|n_{N}} \psi_{n_{1}|...|n_{k-1}|b} n_{k+1|...|n_{N}} \]  

The corresponding expression for an anti-factorist system, individuated by the single-system projector \( E = E_{a}^{b} \) is

\[ \rho(E)_{a} := \frac{E_{m}^{a}}{\Sigma(N)} \left( \psi_{m|...|n_{N}} \psi_{p|...|n_{N}} \right) E_{m}^{b} \]  

where \( \psi_{m|...|n_{N}} \) and we use the fact that \( \psi[a_{1}a_{2}|...|a_{N}] = \psi[a_{1}|a_{2}|...|a_{N}] \), etc.

A comparison between a reduced density operator \( \rho_{1} \), obtained by a partial trace, and one obtained through qualitative individuation by the criterion \( E, \rho(E) \), in Penrose’s graphical tensor notation is given in the top of Figure 4.

Turning to the individuation of multiple systems at once, if we use the single-system individuation criteria \( E_{1},...,E_{N} \), then we first define the projector  

\[ E_{b_{1}|...|b_{N}} := N! \Sigma(N) \left( (E_{1})_{b_{1}}|...|(E_{N})_{a_{N}} \right) ; \]  

the general form of the joint density operator under “Strawsonian” qualitative individuation (see section 3.5), is then given by

\[ \rho(E_{1},...,E_{N})_{a_{1}|...|a_{N}} := \frac{E_{m_{1}|...|m_{N}} \psi_{m_{1}|...|m_{N}} \psi_{n_{1}|...|n_{N}} E_{b_{1}|...|b_{N}}}{E_{m_{1}|...|m_{N}} \psi_{m_{1}|...|m_{N}} \psi_{n_{1}|...|n_{N}}} \]  

where we use the (anti-) symmetry of \( \psi \). This is displayed in Penrose’s graphical tensor notation in the bottom of Figure 4.

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Figure 4: A comparison between the form of a reduced density operator $\rho_1$ obtained by a partial trace and one $\rho(E)$ obtained by qualitative individuation by the projector $E$, for an arbitrary $N$-system state $\Psi$. The bottom diagram shows the general form of the joint density operator under “Strawsonian” individuation with the individuation criteria $E_1, E_2, \ldots, E_N$. The diagram top-right defines the $N$-system symmetrizer $\Sigma^{(N)}$.

4.4 *Qualitative individuation in Fock space

Expectation values and reduced density operators for Fock space. Fock space is just the direct sum of the $N$-system Hilbert spaces we have been separately considering:

$$F^{\pm}(H) := \bigoplus_{N=0}^{\infty} S^{(N)}_{\pm}(\bigotimes_{N=0}^{N} H) =: \bigoplus_{N=0}^{\infty} S^{(N)}_{\pm}$$

(58)

(where $S^{(N)}_{\pm}$ is the $N$-system (anti-) symmetrization operator, where ‘+’ corresponds to bosons, i.e. symmetrization, and ‘−’ to fermions, i.e. anti-symmetrization). The generalization to Fock space is straightforward, which vindicates the claim that anti-factorism about quantum mechanics meshes with quantum field theory in the limit of conserved total particle number. We already have a prescription for calculating expectation values and reduced density operators for a system individuated by any criterion $E_\alpha$ in a joint Hilbert space of $N$ systems. The strategy between the Fock space generalizations of these prescriptions is simply to act separately according to them on subspaces of Fock
space characterized by total particle number $N$.

First we define the “number operator” on Fock space as

$$n_\alpha := \bigoplus_{N=0}^\infty n^{(N)}_\alpha$$

(59)

$$= \bigoplus_{N=0}^\infty \left( \sum_{k=1}^{N-k} \mathbb{1} \otimes E_\alpha \otimes \mathbb{1} \right)$$

(60)

The appropriate function from the single-system algebra to the Fock space algebra $\pi_\alpha^\pm : B(\mathcal{H}) \to B(\mathcal{F}_\pm(\mathcal{H}))$ is then defined by its action on an arbitrary single-system operator $Q$:

$$\pi_\alpha^\pm(Q) = \bigoplus_{N=0}^\infty \left( \sum_{k=1}^{N-k} \mathbb{1} \otimes E_\alpha Q E_\alpha \otimes \mathbb{1} \right)$$

(61)

and we have $\pi_\alpha^\pm(\mathbb{1}) = n_\alpha$, as expected. and Expectation values for any single-system beable $A$ are given by $\langle Q \rangle_\alpha = (\pi_\alpha^\pm(Q))$. While the single-system expectation operator $\langle \cdot \rangle_\alpha : D(\mathcal{H}) \times B(\mathcal{H}) \to \mathbb{C}$ is as usual, now $\langle \cdot \rangle : D(\mathcal{F}_\pm(\mathcal{H})) \times B(\mathcal{F}_\pm(\mathcal{H})) \to \mathbb{C}$ is the expectation operation on density operators and bounded operators on Fock space.

Connections to the Fock space formalism. It will be instructive to connect the expressions above to the more familiar aspects of the Fock space formalism, viz. particle creation, annihilation and numbers operators. Recall that we may define an annihilation operator $a$ valued function $\pi_\alpha^\pm$ and we have

$$A^{(n)}_\alpha \rightarrow B^{(\mathcal{F}_\pm(\mathcal{H}))}$$

(58)

Then the action of the number operator $\hat{N}(\phi) := a^\dagger(\phi) a(\phi)$ is

$$\hat{N}(\phi) \Psi = 0 \otimes \xi_1 \phi n \psi_n^{(1)} \otimes \sqrt{2} \xi_2 \phi n \psi_n^{(2)} \otimes \ldots \otimes \sqrt{N} \xi_N \phi n \psi_n^{(N)} \psi_1^{a_1\ldots a_N} \Psi \psi_2^{a_2\ldots a_N} \otimes \ldots \psi_N^{a_1\ldots a_N} \otimes \ldots$$

(64)

Then the action of the number operator $\hat{N}(\phi) := a^\dagger(\phi) a(\phi)$ is

$$\hat{N}(\phi) \Psi = 0 \otimes \xi_1 \phi n \psi_n^{(1)} \otimes 2 \xi_2 \phi n \psi_n^{(2)} \otimes \ldots \otimes \sqrt{N} \xi_N \phi n \psi_n^{(N)} \psi_1^{a_1\ldots a_N} \Psi \psi_2^{a_2\ldots a_N} \otimes \ldots \psi_N^{a_1\ldots a_N} \psi_1^{a_1\ldots a_N} \otimes \ldots \psi_N^{a_1\ldots a_N} \otimes \ldots$$

(65)
These definitions, combined with the (anti-) symmetry of Ψ, entail the identity

\[ \hat{N}(\phi) \equiv \bigoplus_{N=0}^{\infty} \left( \sum_{k=1}^{N} \otimes 1 \otimes |\phi\rangle \langle \phi| \otimes \otimes 1 \right) \delta_{\pm}^{(N)} \]  

(Proof sketch: We consider separately the action of \( \hat{N}(\phi) \) on each \( N \)-system component \( \Psi|_{\delta_{\pm}^{(N)}} \) of an arbitrary state \( \Psi \in \mathcal{F}_{\pm}(\mathcal{H}) \), and demonstrate the identity for each component. Consider for example the \( N = 2 \) component. From (65) we have \( \hat{N}(\phi)\Psi|_{\delta_{\pm}^{(2)}} = 2\xi_2 \phi^{\alpha}(\chi)^{m} |n| \pm = \xi_2 (\phi^{a} \phi^{n} |n| \pm \pm \phi^{b} \phi^{n} |n| \pm) \). But due to the (anti-) symmetry of \( \Psi \), \( \psi^{\alpha} \equiv \pm \psi^{mn} \), so we have \( \hat{N}(\phi)\Psi|_{\delta_{\pm}^{(2)}} = \xi_2 (\phi^{a} \phi^{n} \psi^{mn} + \phi^{b} \phi^{n} \psi^{mn}) \equiv \xi_2 (\phi^{a} \phi^{n} \delta^{m}_{n} + \delta^{m}_{n} \phi^{b} \phi^{n}) \psi^{mn} \equiv (|\phi\rangle \langle \phi| \otimes 1 + 1 \otimes |\phi\rangle \langle \phi|)\Psi|_{\delta_{\pm}^{(2)}}, \) in agreement with (66).) It may also be checked that the usual (anti-) commutation relations hold, for all \( \phi, \chi \in \mathcal{H} \):

\[ [a(\phi), a^\dagger(\chi)]_\mp = [a^\dagger(\phi), a(\chi)]_\mp = 0; \quad [a(\phi), a^\dagger(\chi)]_\mp = \phi n \chi^m \equiv \langle \phi| \chi \rangle, \]  

where ‘−’ now corresponds to bosons (commutation) and ‘+’ to fermions (anti-commutation).

Now select some orthonormal basis \( \{|\phi_{i}\rangle\}; i = 1, \ldots, d_{\alpha} \) ∈ dim(\( E_{\alpha} \)) that spans the space \( E_{\alpha}[\mathcal{H}] \). Then \( E_{\alpha} = \sum_{i=1}^{d_{\alpha}} |\phi_{i}\rangle \langle \phi_{i}| \) and (combining (60) and (66)),

\[ n_{\alpha} = \sum_{i=1}^{d_{\alpha}} \hat{N}(\phi_{i}) \]  

This justifies our calling \( n_{\alpha} \) a “number operator” in the first place—in fact \( n_{\alpha} \) is just a generalisation of the usual Fock space number operators, in which we are typically less than maximally specific about the single-system state whose occupancy we are counting. If we set \( E_{\alpha} = |\alpha\rangle \langle \alpha| \) for some single-particle state \( \alpha \in \mathcal{H} \) (the limit of maximum specificity), then \( n_{\alpha} = \hat{N}(\alpha) \). If we set \( E_{\alpha} = 1 \) (the limit of maximum non-specificity), then \( n_{\alpha} = \hat{N} \), the total number operator, defined by

\[ \hat{N} := 0 \oplus 1_{\mathcal{H}} \oplus 2 \otimes \delta_{\pm}^{(2)} \oplus \ldots \oplus N \otimes \delta_{\pm}^{(N)} \oplus \ldots \]  

Thus \( n_{\alpha} \) determines a spectrum of number operators, reflecting the spectrum of specificity of our chosen individuation criterion \( E_{\alpha} \), parameterized by its dimension \( d_{\alpha} \), with the familiar state number operators and total number operator at opposite extremes.

The function \( \pi_{\alpha}^{\pm}(Q) \) may be re-expressed as follows:

\[ \pi_{\alpha}^{\pm}(Q) = \bigoplus_{N=0}^{\infty} \left[ \sum_{k=1}^{N} \otimes 1 \otimes |\phi_{i}\rangle \langle \phi_{i}| \right] Q \left( \sum_{j=1}^{d_{\alpha}} |\phi_{j}\rangle \langle \phi_{j}| \right) \otimes \otimes 1 \right| \delta_{\pm}^{(N)} \]  

\[ = \sum_{i,j=1}^{d_{\alpha}} (|\phi_{i}\rangle \langle \phi_{j}|) \bigoplus_{N=0}^{\infty} \left( \sum_{k=1}^{N} \otimes 1 \otimes |\phi_{i}\rangle \langle \phi_{j}| \otimes \otimes 1 \right) \right| \delta_{\pm}^{(N)} \]  

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We now make use of the identity
\[
a^\dagger(\phi) a(\chi) \equiv \bigoplus_{N=0}^{\infty} \left( \sum_{k=1}^{N} \bigotimes_{l=1}^{N-k} 1 \otimes | \phi \rangle \langle \chi | \otimes \bigotimes_{l=1}^{k} 1 \right) \big|_{S_{(N)}^k}\]  
(72)

(which may be verified, as with $\hat{N}(\phi)$ above, by considering the action on each $N$-system component of an arbitrary Fock space state) to re-express (71) as

\[
\pi_{\alpha}^\pm(Q) = \sum_{i,j=1}^{d_\alpha} \langle \phi_i | Q | \phi_j \rangle \ a^\dagger(\phi_i) a(\phi_j)
\]  
(73)

And the general expression for the expectation value of $Q$ for the $\alpha$-system is

\[
\langle Q \rangle_{\alpha} = \frac{\sum_{i,j=1}^{d_\alpha} \left\langle a^\dagger(\phi_i) a(\phi_j) \right\rangle \langle \phi_i | Q | \phi_j \rangle}{\sum_{i=1}^{d_\alpha} \left\langle a^\dagger(\phi_i) a(\phi_i) \right\rangle}
\]  
(74)

If $Q$ has the $|\phi_i\rangle$ as eigenstates ($Q | \phi_i\rangle = q_i | \phi_i\rangle$)—which may always be arranged if $Q$ commutes with the individuation criterion; $[Q, E_{\alpha}] = 0$—then (74) simplifies to the reassuring expression

\[
\langle Q \rangle_{\alpha} = \frac{\sum_{i=1}^{d_\alpha} \langle \hat{N}(\phi_i) \rangle q_i}{\sum_{i=1}^{d_\alpha} \langle \hat{N}(\phi_i) \rangle}
\]  
(75)

which has a straightforward physical interpretation as being the average eigenvalue for $A$ of all Fock space quanta whose state satisfies the individuation criterion $E_{\alpha}$. If the single-system operator and individuation criterion do not commute ($[Q, E_{\alpha}] \neq 0$), then the cross-terms in (73) survive, and will manifest experimentally as interference between the single-system states $|\phi_i\rangle$.

A reduced density operator associated with the individuation criterion $E_{\alpha}$ can also be defined for Fock space, so that $\langle Q \rangle_{\alpha} = \text{Tr}(\rho_{\alpha} Q)$, for all $Q \in \mathcal{B}(\mathcal{H})$. Given an orthonormal basis \{|$\phi_i$\}\ for $\mathcal{H}$, we have (cf. (36), above),

\[
\rho_{\alpha} = \sum_{i,j}^{d} |\phi_i\rangle \langle \phi_j | \langle \phi_j | \langle \phi_i\rangle_{\alpha}
\]  
(76)

We can arrange for the first $d_\alpha$ basis states to span $E_{\alpha}[\mathcal{H}]$, in which case

\[
\rho_{\alpha} = \frac{\sum_{i,j}^{d_\alpha} \left\langle a^\dagger(\phi_i) a(\phi_j) \right\rangle | \phi_i\rangle \langle \phi_j |}{\sum_{i=1}^{d_\alpha} \left\langle a^\dagger(\phi_i) a(\phi_i) \right\rangle}
\]  
(77)

which is clearly a positive operator (since $\hat{N}(\phi_i)$ is, for all $\phi_i \in \mathcal{H}$) with unit trace.
4.5 Ubiquitous and unique systems

In section 4.1 above, I deliberately defined a procedure for calculating expectation values for qualitatively individuated systems that does not depend on the joint state being an \( n_\alpha = 1 \) eigenstate. This has advantages for purposes of generality, but individuation criteria \( E_\alpha \) for which the joint state is an \( n_\alpha = 1 \) eigenstate, and their associated systems, are also extremely useful. This is because they are anti-factorist surrogates for “distinguishable” systems, which are identifiable in all states by the same factor Hilbert space label—and their individuation criteria are the anti-factorist surrogates for factor Hilbert space labels. I will call such systems ubiquitous and unique: ubiquitous, because some single-particle state in \( E_\alpha[H] \) is occupied in each term of the joint state; unique, because such states occur only once in each such term.

Ubiquitous and unique systems, like “distinguishable” systems, probe the entire joint state, and never appear more than once; so their reduced states give valuable information about the joint state. In particular, they will be helpful in the next section (5.4) in providing a means to define a continuous measure of entanglement for the joint state.

If the \( \alpha \)-system is ubiquitous and unique, then we have by definition \( \text{Tr}(\rho_{n\alpha}) = 1 \) for the joint state \( \rho \). But \( n_\alpha = N\Sigma^{(N)}(E_\alpha \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}) \), so

\[
\text{Tr}(\rho_{n\alpha}) = 1 = N\text{Tr} \left[ \rho\Sigma^{(N)} \left( E_\alpha \otimes \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{N-1} \right) \right]
\]

(78)

\[
= \frac{1}{(N-1)!} \sum_{\pi \in S_N} \text{Tr} \left[ \rho P(\pi) \left( E_\alpha \otimes \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{N-1} \right) P^\dagger(\pi) \right]
\]

(79)

\[
= N\text{Tr} \left[ \rho \left( E_\alpha \otimes \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{N-1} \right) \right]
\]

(80)

\[
= N\text{tr} (\rho_1 E_\alpha)
\]

(81)

\[
= N\text{tr} (\bar{\rho} E_\alpha)
\]

(82)

where in the third step we use the cyclicity of the trace function and the fact that \( P(\pi)\rho P^\dagger(\pi) = \rho \), due to PI; in the fourth step we use the definition of the partial trace; and in the fifth step we use the fact that \( \rho_1 = \bar{\rho} \) when PI is imposed. Thus for ubiquitous and unique systems individuated by \( E_\alpha \),

\[
\text{tr}(\bar{\rho} E_\alpha) = \frac{1}{N}
\]

(83)

and the expression (51) for the reduced density operator for “the” \( \alpha \)-system simplifies to

\[
\rho_\alpha = N E_\alpha \bar{\rho} E_\alpha.
\]

(84)

This has the following mathematical interpretation. If \( \bar{\rho} \) is written in matrix form, in a basis such that the first \( d_\alpha \) states span \( E_\alpha[H] \), then the top-left block of this matrix—given by \( E_\alpha \bar{\rho} E_\alpha \)—is the matrix \( \frac{1}{N} \rho_\alpha \).
Similar results follow for the $\beta$-system, and so on. So if a full $N$ ubiquitous and unique systems (with mutually orthogonal individuation criteria) can be found for the entire joint state—which will be the case iff the joint state lies in an individuation block—then $N\bar{\rho}$ can be put into a block-diagonal form with all of the qualitatively individuated systems’ states as the diagonal entries:

$$N\bar{\rho} = \begin{pmatrix} \rho_\alpha & \rho_\beta \\ \rho_\beta & \ddots \end{pmatrix} \quad (85)$$

(where there are $N$ diagonal blocks). We see here quite vividly that $\bar{\rho}$ represents the “average state” of the constituent systems. This result will come in useful in the next section.

5 Entanglement

In this section I draw out the implications of anti-factorism for the notion of entanglement. First (section 5.1) I say why the traditional definition, in terms of non-separability, needs supplanting. Then in section 5.2, I will connect these considerations with recent heterodox proposals for defining entanglement, or “quantum correlations”, in the context of permutation invariance, and I present two results which secure the suitability of the agreed definition, and in section 5.4 offer a continuous measure for the notion; it will turn out to have a simple relation to the familiar von Neumann entropy of the reduced state.

5.1 Against non-separability

If anti-factorism is right, then nothing in the formalism with physical significance can depend on the ordering of the factor Hilbert spaces in the joint Hilbert space. But this immediately compels a distinction between two kinds of non-separability that a joint state may exhibit.

Separability, I take it, is a purely formal notion (at least, I will here take it to be): an $N$-system joint state $|\Psi\rangle$ is separable iff there is a basis of the single-system Hilbert space $\mathcal{H}$ for which $|\Psi\rangle$ is some $N$-fold tensor product of those basis states. But a joint state may fail to be separable purely because it is a superposition of product states which differ only by how single-system states are distributed among the factor Hilbert spaces (e.g. $|\phi\rangle \otimes |\chi\rangle$ and $|\chi\rangle \otimes |\phi\rangle$). Under the superselection rule induced by PI, the only such (pure) joint states are of the form

$$|\psi^\pm (\phi)\rangle = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (\pm 1)^{\deg(\pi)} P(\pi)|\phi(1)\rangle \otimes |\phi(2)\rangle \otimes \ldots \otimes |\phi(N)\rangle \quad (86)$$
where \( \phi : \{1, \ldots, N\} \to \{1, \ldots, d\} \) and \( \{|1\rangle, \ldots, |d\rangle\} \) is an orthonormal basis for \( \mathcal{H} \) (so \( \phi \) may be thought of as a function that takes factor Hilbert space labels to single-system states). In the case of bosons we allow \( \phi \) not to be injective, but in that case the state in (86) is not properly normalized.

If \( \phi \) is injective, then the joint state \(|\psi^\pm(\phi)\rangle\) lies in an individuation block in \( \mathfrak{S}_2^{(N)} \)—in fact, it lies in a \( 1 \)-dimensional individuation block. The corresponding \( N \) individuation criteria, which are \( 1 \)-dimensional, are \( E_i = |\phi(i)\rangle \langle \phi(i)|, \ i = 1, 2, \ldots N \). Following our interpretative principle (above, section 2.4) that intertwiners between unitarily equivalent reps preserve physical interpretations, we are compelled to give the same interpretation to \(|\psi^\pm(\phi)\rangle\) that we give to each of the product states \( P(\pi) |\phi(1)\rangle \otimes \cdots \otimes |\phi(N)\rangle \) when PI is not imposed. But in these states, being separable, are unentangled. It follows that, if entanglement is to be a \textit{physical}, rather than formal, concept, then we ought also to construe states of the form (86) as unentangled. (I have overlooked what to say about joint states of that form in which \( \phi \) is not injective: this will be remedied in the following section.)

Some readers may baulk at this suggestion. After all, the spin singlet state for two qubits,

\[
\frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle),
\]

(87)

has the form (86), yet this is the most commonly mentioned state in discussions of entanglement and in derivations of Bell-type inequalities (e.g. Bell (1976))! How could the paradigm of an entangled state \textit{not} count as entangled on a “physical” understanding of that concept?

The resolution to this little puzzle is simply that we need to be more careful about whether PI is or isn’t being imposed. In almost all textbook treatments, PI is not imposed and one derives a Bell inequality which the singlet state violates. The key beables in that exercise are of the form \( \sigma_i \otimes \mathds{1} \) or \( \mathds{1} \otimes \sigma_j \) (and constructions out of these, chiefly \( \sigma_i \otimes \sigma_j \)), which manifestly make non-trivial use of the ordering of the factor Hilbert spaces, and therefore do not satisfy PI. These beables belong to an algebra of operators defined on the \textit{full} joint Hilbert space \( \mathfrak{S}_2^{(2)} = \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \). In short: the constituent systems are treated as “distinguishable”.

If the systems are treated as “indistinguishable”—i.e., when PI is imposed—then the singlet state constitutes the \textit{only} state accessible to two fermions; i.e. \( \mathfrak{S}_2^{(2)} = \mathbb{C} \). The corresponding joint algebra is \( \mathcal{B}(\mathbb{C}) \cong \mathbb{C}, \) i.e. only \( c \)-numbers, which is far too meagre to accommodate the violation of a Bell inequality—it is too meagre even to define spin operators for each of the constituent systems (construed factoristically)!

In fact we know that the real-world systems whose spin-measurements violate Bell inequalities do not only possess spin as a degree of freedom. (This point and its implications are well put by Ladyman \textit{et al} (2013, 216).) If we are pedantic about this, we ought also to include the spatial degrees of freedom. (After all, Stern-Gerlach apparatuses exist...
in space! Let us therefore consider the following four-component state:
\[ \frac{1}{2} (|L\rangle_1 \otimes |R\rangle_2 + |R\rangle_1 \otimes |L\rangle_2) \otimes (|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2), \] (88)
(where I have used labels to keep track of the single-system factor Hilbert spaces). This state lies in the fermionic joint Hilbert space, and is not of the form (86), so it counts as entangled even according to the current heterodox proposal. Accordingly, it may be shown that it yields the same Bell-inequality-violating spin measurements in the context of PI as the singlet state does when PI is not imposed.

The equivalence is demonstrated by appealing to qualitative individuation. We choose our individuation criteria for the two constituent systems to be \( E_L := |L\rangle \langle L| \otimes 1_{\text{spin}} \) and \( E_R := |R\rangle \langle R| \otimes 1_{\text{spin}} \). Then it may be checked that, by (9), the operators in the joint algebra corresponding to the spins (in units of \( \frac{1}{2} \hbar \)) of the systems qualitatively individuated by \( E_L \) and \( E_R \) are

\[
\sigma_i(L) := U_-(\sigma_i \otimes 1)U_+^\dagger = (|L\rangle \langle L| \otimes \sigma_i) \otimes (|R\rangle \langle R| \otimes 1_{\text{spin}}) + (|R\rangle \langle R| \otimes 1_{\text{spin}}) \otimes (|L\rangle \langle L| \otimes \sigma_i),
\] (89)

\[
\sigma_i(R) := U_-(1 \otimes \sigma_i)U_+^\dagger = (|R\rangle \langle R| \otimes \sigma_i) \otimes (|L\rangle \langle L| \otimes 1_{\text{spin}}) + (|L\rangle \langle L| \otimes 1_{\text{spin}}) \otimes (|R\rangle \langle R| \otimes \sigma_i),
\] (90)

which, given the state (88), yield the spin-correlation expectation values

\[
\langle u^i \sigma_i(L)v^j \sigma_j(R) \rangle = -u_i v^j
\] (91)

for any two unit spatial vectors \( u^i \) and \( v^i \) (where summation over repeated indices is implied). This suffices to violate the Clauser-Horne-Shimony-Holt (CHSH) inequality

\[
\left| \langle u^1_1 \sigma_1(L)v^1_2 \sigma_2(R) \rangle + \langle u^1_1 \sigma_1(L)v^2_2 \sigma_2(R) \rangle + \langle u^2_2 \sigma_1(L)v^1_2 \sigma_2(R) \rangle - \langle u^2_1 \sigma_1(L)v^2_2 \sigma_2(R) \rangle \right| \leq 2
\] (92)

when suitable choices for \( u^1_1, v^1_1, u^1_2, v^2_2 \) are made (e.g. in the \( x-y \) plane: \( u_1 = (0, 1); u_2 = (1, 0); v_1 = \frac{1}{\sqrt{2}}(1, 1); v_2 = \frac{1}{\sqrt{2}}(-1, 1) \)).

The singlet state, construed such that PI has not been imposed, is frequently used as a simplifying surrogate for states of the form (88), for which PI has been imposed. (If your interest is in discussing entanglement, why complicate matters by worrying about permutation invariance?) That the singlet does serve as a suitable simplifying surrogate is justified by appeal to the fact that it yields the spin-correlation expectation values

\[
\langle u^i \sigma_i \otimes v^j \sigma_j \rangle = -u_i v^j,
\] (93)

in agreement with (91).

It may also be of interest that the prescription (36) yields the following reduced density operators for the state (88):

\[
\rho_L = |L\rangle \langle L| \otimes \frac{1}{2} 1_{\text{spin}}, \quad \rho_R = |R\rangle \langle R| \otimes \frac{1}{2} 1_{\text{spin}}\] (94)
If we then trace out the spatial states, we obtain the density operator $\frac{1}{2}I_{\text{spin}}$ for each qualitatively individuated system—which is normally associated with the (factoristically construed!) constituent systems of the singlet state when IP is not imposed.

Finally, I note that the state

$$\frac{1}{\sqrt{2}}(|L_1\rangle \otimes |\uparrow_1\rangle \otimes |R_2\rangle \otimes |\downarrow_2\rangle - |R_1\rangle \otimes |\downarrow_1\rangle \otimes |L_2\rangle \otimes |\uparrow_2\rangle) \quad (95)$$

which is of the form (86), and therefore according to the current proposal ought not to count as entangled, does not violate the CHSH inequality (92) for any choice of $u_1^i, v_1^i, u_2^i, v_2^i$, and gives the reduced density operators $\rho_L = |L\rangle\langle L| \otimes |\uparrow\rangle\langle \uparrow|$, $\rho_R = |R\rangle\langle R| \otimes |\downarrow\rangle\langle \downarrow|$ (i.e. pure states) for the constituent systems: two features which, non-separability notwithstanding, we normally associate with the absence of entanglement.

### 5.2 Connections to the physics literature I: GMW-entanglement

The above considerations are found in a series of papers by Ghirardi, Marinatto and Weber (Ghirardi, Marinatto & Weber (2002); Ghirardi & Marinatto (2003, 2004, 2005)). They argue that the usual definition of entanglement (non-separability) can be naturally adapted to quantum systems governed by PI in the way just shown. To avoid confusion with non-separability, I will call their adapted definition, and the definition I wish here to endorse, GMW-entanglement.

For simplicity, I assume $N = 2$; generalizations for $N > 2$ will be obvious. Following Ghirardi, Marinatto and Weber (2002), define:

The indistinguishable constituents of a two-system assembly are non-GMW-entangled (we may say the joint state is non-GMW-entangled) iff both systems have a complete set of properties.

(The assembly is then defined as GMW-entangled iff it is not non-GMW-entangled.) And we define:

Given a joint state $\rho$ of two “indistinguishable” systems, at least one of the systems has a complete set of properties iff there exists a 1-dimensional projector $E$, defined on $\mathcal{H}$, such that:

$$\text{Tr}(\rho E) = 1 \quad (96)$$

where

$$E := E \otimes 1 + 1 \otimes E - E \otimes E. \quad (97)$$

Because $E$ is 1-dimensional, Ghirardi and Marinatto say that at least one of the systems has a ‘complete set of properties’. ‘Complete’ here picks up on the fact that $E$ is minimal, and therefore maximally specific (maximally logically strong) about the system’s state.
This almost corresponds above to the two systems being qualitatively individuated by the projectors $E$ and $E^\perp := (\mathbb{1} - E)$. The difference is that $\mathcal{E}$ also projects onto doubly-occupied joint states (which must be bosonic); thus we say ‘at least one system has a complete set of properties’.

Ghirardi and Marinatto (2003, Theorems 4.2 & 4.3) then prove the following theorem:

At least one of the systems in a two-system assembly has a complete set of properties iff the assembly’s state is obtained by symmetrizing or anti-symmetrizing a separable state.

Here I give a quick sketch of their proof:

**Right to left (easy half):** If $|\Psi\rangle$ is obtained by symmetrizing or anti-symmetrizing a factorized state of two indistinguishable constituents, then:

$$|\Psi\rangle = \frac{1}{\sqrt{2(1 \pm |\langle \phi|\chi \rangle|^2)}} (|\phi \rangle \otimes |\chi \rangle \pm |\chi \rangle \otimes |\phi \rangle).$$  \hspace{1cm} (98)

By expressing the state $|\chi \rangle$ as

$$|\chi \rangle = \alpha|\phi \rangle + \beta|\phi^\perp \rangle,$$

and choosing $E = |\phi \rangle \langle \phi |$, one gets immediately

$$\text{Tr}(\rho E) \equiv |\langle \Psi\pm |E|\Psi\pm \rangle| = 1.$$  \hspace{1cm} (100)

**Left to Right (hard half):** If one chooses a complete orthonormal set of single-system states whose first element $|\phi_0 \rangle := |\phi \rangle$ is such that $E = |\phi_0 \rangle \langle \phi_0 |$, writing

$$|\Psi\rangle = \sum_{ij} c_{ij} |\phi_i \rangle \otimes |\phi_j \rangle,$$

$$\sum_{ij} |c_{ij}|^2 = 1, \hspace{0.5cm} c_{ji} = \pm c_{ij},$$  \hspace{1cm} (101)

and, using the explicit expression for $\mathcal{E}$ in terms of $E$, one obtains:

$$\mathcal{E}|\Psi\rangle = |\phi_0 \rangle \otimes \left( \sum_{j \neq 0} c_{0j} |\phi_j \rangle \right) + \left( \sum_{j \neq 0} c_{j0} |\phi_j \rangle \right) \otimes |\phi_0 \rangle + c_{00} |\phi_0 \rangle \otimes |\phi_0 \rangle.$$  \hspace{1cm} (102)

But now imposing condition (96) entails that $\mathcal{E}|\Psi\rangle = |\Psi\rangle$.

In the case of fermions, $c_{00} = 0$. So, introducing a normalized vector $|\xi \rangle := \sqrt{2 \sum_{j \neq 0} c_{0j} |\phi_j \rangle}$, we obtain

$$|\Psi_- \rangle = \frac{1}{\sqrt{2}} \left( |\phi_0 \rangle \otimes |\xi \rangle - |\xi \rangle \otimes |\phi_0 \rangle \right),$$  \hspace{1cm} (103)

where $|\langle \phi_0 |\xi \rangle| = 0$. For bosons, defining the following normalized vector

$$|\theta \rangle = \frac{1}{\sqrt{2 - |\langle c_{00} |^2}}} \left( \sum_{j \neq 0} 2c_{0j} |\phi_j \rangle + c_{00} |\phi_0 \rangle \right),$$  \hspace{1cm} (104)

44
the two-particle state vector becomes
\[
|\Psi_+\rangle = \frac{1}{2}\sqrt{2-|c_{00}|^2} \left( |\phi_0\rangle \otimes |\theta\rangle + |\theta\rangle \otimes |\phi_0\rangle \right).
\] (105)

Note that in this case the states \(|\phi_0\rangle\) and \(|\theta\rangle\) are orthogonal iff \(c_{00} = 0\), in which case \(|\theta\rangle = |\xi\rangle\).

Let us now consider GMW-entanglement specifically for fermions and bosons, respectively.

**GMW-entanglement for fermions.** Since \(E \otimes E = 0\) on \(H^{(2)}\), one can drop such a term in all previous formulae. Accordingly, \(E_f := E \otimes 1 + 1 \otimes E\).

Due to the orthogonality of \(|\phi_0\rangle\) and \(|\xi\rangle\), for the state (103), we conclude not only that there is one fermion with a complete set of properties (given by \(|\phi_0\rangle\)), but also that the other fermion has a complete set of properties (given by \(|\xi\rangle\)).

So according to the definition of GMW-entanglement, we have proved:

The fermions of a two-system assembly, whose (pure) state is given by \(|\Psi_-\rangle\), are non-GM-entangled iff \(|\Psi_-\rangle\) is obtained by anti-symmetrizing a separable state.

**GMW-entanglement for bosons.** The broad similarity to fermions is clear, especially from Equations (98) and (105). As for fermions, the requirement that one of the two bosons has a complete set of properties entails that the state is obtained by symmetrizing a separable state. However, there are some remarkable differences from the fermion case. For bosons, three cases are possible, according to the single-system states that are the factors of the separable state:

1. \(|\theta\rangle \propto |\phi_0\rangle\), i.e. \(|c_{00}| = 1\). Then the state is \(|\Psi_+\rangle = |\phi_0\rangle \otimes |\phi_0\rangle\) and one can infer that there are two bosons each with the same complete set of properties given by \(E = |\phi_0\rangle \langle \phi_0|\). It may checked that for this state \(\langle \Psi_+|E \otimes E|\Psi_+\rangle = 1\). \(|\Psi\rangle\) belongs to no individuation block.

2. \(|\theta\rangle \propto |\phi_0\rangle\), i.e. \(c_{00} = 0\). Then exactly the same argument as for fermions entails that one of the two bosons has a complete set of properties given by \(E := |\phi_0\rangle \langle \phi_0|\) and the other of the two bosons has a complete set of properties given by \(F := |\theta\rangle \langle \theta|\). The joint state \(|\Psi_+\rangle\) belongs the the 1-dimensional individuation block defined by these two single-system states.

3. Finally, it can happen that \(0 < |\langle \theta|\phi_0\rangle| < 1\). In this case, it is true from the definition above that there is a boson with a complete set of properties given by \(E\) and it is true that there is a boson with a complete set of properties given by \(F\). But it is not true that one has a complete set of properties given by \(E\) and the other has a complete set of properties given by \(F\). For there is a non vanishing probability of finding both particles in the same state, since \(\langle \Psi_+|E \otimes F|\Psi_+\rangle = |c_{00}|^2 > 0\).
According to our definition of GMW-entanglement, the joint state is non-GMW-entangled for the first two cases. But in the last case we cannot say that the joint state is non-GMW-entangled, even though we may say that one system has a complete set of properties given by $E$ and one system has a complete set of properties given by $F$. The worry, of course, is that we are counting contributions from the same system each time, so we must resist the plural article ‘both’. So any state of this third type counts as GMW-entangled, being a superposition of states of the first and second types. To sum up:

The bosons of a two-system assembly, whose (pure) state is given by $|\Psi_+\rangle$, are non-GM-entangled iff either: (i) $|\Psi_+\rangle$ is obtained by symmetrizing a factorized product of two orthogonal states; or (ii) $|\Psi_+\rangle$ is a product state of identical factors.

To connect with our usual terminology: a fermionic joint state is GMW-entangled iff it lies in a 1-dimensional individuation block. A bosonic state is GMW-entangled iff: (i) it lies in a 1-dimensional individuation block (case 1, above); or (ii) it is a product state of identical factors. For $N > 2$, we must add a third clause: (iii) all sub-assemblies satisfy either (i) or (ii). If the joint state lies in an individuation block, for both bosons and fermions the intertwiner between the individuation block and its corresponding product Hilbert space takes GMW-entanglement in the “indistinguishable” case to entanglement in the “distinguishable” case.

This is to be compared to the fact regarding “distinguishable” systems, that their joint state is non-entangled iff the constituent systems (associated with factor Hilbert spaces) occupy pure states. Both results have the same physical interpretation: that in non-(GMW-)entangled states, the joint state of the assembly supervenes on (i.e. is uniquely determined by) the states of its constituent systems. This is a good reason to take GMW-entanglement as the right notion of entanglement when PI is imposed.

A further reason is given by the following result concerning Bell inequalities. From Gisin (1991), we know that, for two “distinguishable” systems, any joint state of theirs is entangled iff it violates a Bell inequality for some beables. But we have seen that any 2-system individuation block is unitarily equivalent to some product Hilbert space for 2 systems, and that the intertwiner between them takes entanglement to GMW-entanglement. Therefore we conclude:

If the joint state lies in an individuation block, then: the joint state is GMW-entangled iff it violates a Bell inequality for some symmetric beables.

5.3 Connections to the physics literature II: “quantum correlations”

An alternative definition of entanglement for 2-system assemblies, the existence of quantum correlations, has also been suggested by Schliemann et al (2001) and Eckert et al
(2002). They introduce the notion of the Slater rank of a joint state (Schliemann et al. (2001, 8)), in analogy with the Schmidt rank of a joint state of two “distinguishable” systems.

For any such state, we can carry out a bi-orthogonal Schmidt decomposition:

$$|\Psi\rangle = \sum_i c_i |\phi_i\rangle \otimes |\chi_i\rangle,$$

(106)

where $$\langle \phi_i | \phi_j \rangle = \langle \chi_i | \chi_j \rangle = 0$$ for $$i \neq j$$. The Schmidt rank of $$|\Psi\rangle$$ is the minimum integer $$r$$, given all possible Schmidt decompositions, such that the sum (106) has $$r$$ non-zero entries. Clearly $$|\Psi\rangle$$ is entangled iff $$r > 1$$.

Now consider an arbitrary 2-fermion state,

$$|\Psi-\rangle = \sum_{i<j} c_{ij} \frac{1}{\sqrt{2}} (|\phi_i\rangle \otimes |\phi_j\rangle - |\phi_j\rangle \otimes |\phi_i\rangle) \equiv \sum_{i<j} c_{ij} a^\dagger(\phi_i)a^\dagger(\phi_j)|0\rangle,$$

(107)

where $$|0\rangle$$ is the Fock space vacuum, and $$a^\dagger(\phi_i)$$ is the fermion creation operator for the state $$|\phi_i\rangle$$. What we now need is a fermionic analogue to the bi-orthogonal Schmidt decomposition. This is supplied by the following result from matrix theory (see Schliemann et al. (2001, 3 (Lemma 1)) and Mehta (1977, Theorem 4.3.15)):

For any complex anti-symmetric $$d \times d$$ matrix $$M$$, there is a unitary transformation $$U$$ such that $$\tilde{M} := U M U^T$$ has the form

$$\tilde{M} = \text{diag}[Z_1, \ldots, Z_r, Z_0]$$

(108)

where

$$Z_k := \begin{pmatrix} 0 & z_k \\ -z_k & 0 \end{pmatrix}, \ k \in \{1, \ldots, r\} \quad \text{and} \quad Z_0 := O(d-2r)$$

(109)

where $$O(d-2r)$$ is the $$(d-2r) \times (d-2r)$$ null matrix.

If we set $$M := c_{ij}$$, then by an unitary transformation of the single-system Hilbert space $$\mathcal{H}$$, we can get the joint state $$|\Psi-\rangle$$ in the form

$$|\Psi-\rangle = \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} z_i a^\dagger(\chi_{2i-1})a^\dagger(\chi_{2i})|0\rangle.$$

(110)

(where $$\lfloor x \rfloor$$ is the largest integer no greater than $$x$$). Now the fermionic Slater rank of $$|\Psi-\rangle$$ is the minimum number $$r$$ of non-zero terms in (112). We may say that the joint state displays quantum correlations iff $$r > 0$$.

As Eckert et al. (2002, 12) remark, we do not see quantum correlations for $$\text{dim}(\mathcal{H}) =: d < 4$$. We can use our previous results to see give a physical insight into this claim.
First, we must see that any state of the form (112) lies in some individuation block in $\mathcal{H}_{(2)}$. There is some conventionality here, but the following choice of individuation criteria will do:

$$E_1 := \sum_i |\chi_{2i-1}\rangle\langle\chi_{2i-1}|; \quad E_2 := \sum_i |\chi_{2i}\rangle\langle\chi_{2i}|.$$  \hspace{1cm} (111)

I.e., $E_1$ captures all odd states $|\chi_{2i-1}\rangle$ and $E_2$ captures all even states $|\chi_{2i}\rangle$. We may conclude that any 2-fermion state lies in an individuation block; I will return to this in section 6.2.

Now the individuation block defined by $E_1$ and $E_2$ is unitarily equivalent to the joint Hilbert space $E_1[\mathcal{H}] \otimes E_2[\mathcal{H}]$. From (112) we can see that the (inverse of the) intertwiner connecting these joint spaces sends $|\Psi-\rangle$ to

$$U^\dagger |\Psi-\rangle = \sum_{i=1}^{i,j} z_i |\chi_{2i-1}\rangle \otimes |\chi_{2i}\rangle,$$ \hspace{1cm} (112)

which has the bi-orthogonal Schmidt decomposed form of (106). Thus $|\Psi-\rangle$ exhibits quantum correlations iff $U^\dagger |\Psi-\rangle$ is entangled. We also know that, for fermions, the intertwiner takes entanglement in the product space over to GMW-entanglement in the individuation block. We conclude that any 2-fermion joint state exhibits quantum correlations (in the sense of Eckert et al (2002)) iff it is GMW-entangled.

We can now see why, for two fermions to be exhibit quantum correlations, i.e. be GMW-entangled, the single-system Hilbert space requires at least 4 dimensions. For fermionic GMW-entanglement, the joint state must yield mixed states for both constituent systems. This requires at least 2 states per system; the systems may not share any states, since their individuation criteria must be orthogonal. This yields 4 single-system states in total.

Eckert et al (2002) also offer a definition of bosonic quantum correlations. Consider an arbitrary 2-boson state,

$$|\Psi_+\rangle = \sum_{i<j} c_{ij} \frac{1}{\sqrt{2(1+\delta_{ij})}} (|\phi_i\rangle \otimes |\phi_j\rangle + |\phi_j\rangle \otimes |\phi_i\rangle) \equiv \sum_{i<j} c_{ij} b^\dagger(\phi_i) b^\dagger(\phi_j)|0\rangle,$$ \hspace{1cm} (113)

where $|0\rangle$ is the Fock space vacuum, and $b^\dagger(\phi_i)$ is the boson creation operator for the state $|\phi_i\rangle$. We now appeal to the following result from matrix theory (see Eckert et al (2002, 9-10):

For any complex symmetric $d \times d$ matrix $M$, there is a unitary transformation $U$ such that $M' := UMU^T$ has the form

$$M' = \text{diag}[z_1, \ldots z_r, 0, \ldots, 0].$$ \hspace{1cm} (114)
If we set \( M := c_{ij} \), then by an unitary transformation of the single-system Hilbert space \( \mathcal{H} \), we can get the joint state \( |\Psi_+\rangle \) in the form of a superposition purely of doubly-occupied states:

\[
|\Psi_+\rangle = \sum_{i=1}^{d} z_i |\chi_i\rangle \otimes |\chi_i\rangle.
\]  

Then the **bosonic Slater rank** is minimum number \( r \) of doubly-occupied states with non-zero amplitudes. **Bosonic quantum correlations** correspond to \( r > 1 \).

It would be a mistake to follow Eckert et al (2002, 24) in considering a bosonic Slater rank of more than 1 to clearly indicate anything like entanglement. The best case for scepticism here is that the non-GMW-entangled state.

\[
|\Phi_+\rangle := \frac{1}{\sqrt{2}} (|\phi\rangle \otimes |\chi\rangle + |\chi\rangle \otimes |\phi\rangle)
\]  

may be put into the form

\[
|\Phi_+\rangle = \frac{1}{\sqrt{2}} (|\phi'\rangle \otimes |\phi'\rangle - |\chi'\rangle \otimes |\chi'\rangle)
\]  

where

\[
|\phi'\rangle := \frac{1}{\sqrt{2}} (|\phi\rangle + |\chi\rangle); \quad |\chi'\rangle := \frac{1}{\sqrt{2}} (|\phi\rangle - |\chi\rangle);
\]  

and so has a bosonic Slater rank of 2. Thus we have agreement with Ghirardi et al only for fermions. In view of the unitary equivalence results above (section 3.3), and the fact that intertwiners between individuation blocks and product spaces take GMW-entanglement to entanglement, we ought to retain the GMW definition of entanglement in the case of bosons.

### 5.4 A continuous measure of entanglement for 2-system assemblies

In the “distinguishable” case for assemblies of 2 systems (a.k.a. “bipartite systems”), the entanglement of any pure joint state \( |\Psi\rangle \) is commonly measured by the entropy of entanglement \( E(|\Psi\rangle) \), given by the von Neumann entropy of the reduced density operator of one of the constituent systems:

\[
E(|\Psi\rangle) := S(\rho_1) = -\text{tr}(\rho_1 \log_2 \rho_1),
\]  

where \( \rho_1 \) is obtained by a partial trace, i.e. \( \rho_1 = \text{Tr}_1(|\Psi\rangle\langle\Psi|) \). This measure meets the obvious requirement that it be independent of the choice of constituent system (i.e. \( E(|\Psi\rangle) = S(\rho_1) = S(\rho_2) \), where \( \rho_2 = \text{Tr}_2(|\Psi\rangle\langle\Psi|) \)), and it has the desirable properties that: (i) it takes its minimum value 0 for separable joint states, and (ii) it takes the maximum value \( \log_2 d \) when \( \rho_1 = \frac{1}{d} \mathbf{1} \), where \( d := \dim(\mathcal{H}) \) (see Nielsen & Chuang (2010, 513)).
Due to the possibilities afforded by qualitative individuation, we can also define sensible reduced density operators under PI. Thus we can use the von Neumann entropy of these reduced density operators to provide a new continuous measure of entanglement.

We can use the unitary equivalence results above to export any fact holding for “distinguishable” systems over to “indistinguishable” systems, so long as the joint state lies in an individuation block. In particular, we can export the fact that the von Neumann entropy of the reduced density operator of one of the constituents is equal to that of the other. I.e., $S(\rho_\alpha) = S(\rho_\beta)$. (Note that it is important that systems be ubiquitous and unique.) This fact, together with the form (85) of $\rho$ derived in the last section, we may infer

$$S(\rho) := -\text{Tr}(\rho \log_2 \rho)$$

(120)

$$= -\frac{1}{2} \text{Tr}[2\rho \log_2(2\rho)] + \log_2 2$$

(121)

$$= \frac{1}{2} [\text{Tr}(\rho_\alpha \log_2 \rho_\alpha) + \text{Tr}(\rho_\beta \log_2 \rho_\beta)] + \log_2 2$$

(122)

$$= \frac{1}{2} [S(\rho_\alpha) + S(\rho_\beta)] + \log_2 2$$

(123)

$$= S(\rho_\alpha) + \log_2 2.$$  

(124)

Therefore the von Neumann entropy of any qualitatively individuated system differs from that of a factorist system only by a constant characterised by the total number of systems, 2.

For fermion states, $S(\rho)$ cannot be less than $\log_2 2$. Under a factorist interpretation, this minimum indicates “inaccessible” entanglement brought about by the (anti-) symmetrization dictated by PI. However, under an anti-factorist interpretation, this minimum is a result of Pauli exclusion, combined with the fact that $\rho$ is not the state of any genuine system, but rather a statistical artefact, an “average state”. Given Pauli exclusion, this average state cannot be definite, and is less definite the more fermions there are; thus the entropy of the state cannot fall under $\log_2 2$. On the other hand, the von Neumann entropies $S(\rho_\alpha)$ and $S(\rho_\beta)$ of the genuine systems has a minimum value of zero, corresponding to a non-GMW-entangled joint state for the assembly.

The maximum value of $S(\rho)$ is $\log_2 d$, where $d$ is the dimension of the single-system Hilbert space, corresponding to $\rho = \frac{1}{d}I$. It follows that, whenever the joint state lies in an individuation block, the qualitatively individuated systems cannot have a von Neumann entropy exceeding $\log_2(\frac{d}{2})$. In fact, the results of section 5.3 entail that for fermions this upper limit can be refined to $\log_2(\frac{d}{2})$. Finally, it may also be shown that the entropy of entanglement for a joint state of 2 fermions never exceeds $\log_2 r$, where $r$ is its fermionic Slater rank.

All this suggests that a good measure of GMW-entanglement is given by $E_{GMW}(|\Psi\rangle) := S(\rho) - \log_2 2 \equiv E(|\Psi\rangle) - \log_2 2$. As we have seen, this measure carries over to “indistinguishable” systems all of the desirable properties of the entropy of entanglement for “distinguishable” systems. Its only drawback is that applies only when the joint state
lies in an individuation block. However, as we shall see in the next section, this is a restriction only for bosonic states. How to measure entanglement for bosonic states which do not lie in an individuation block—i.e. those that contain terms with identical factors—is still an open question.

6 Implications for discernibility

6.1 When are “indistinguishable” quantum systems discernible?

As we saw in section 3.3, any individuation block of either the fermion or boson N-system joint Hilbert space “behaves” in every way (restricted to that block), due to unitary equivalence, like a corresponding joint Hilbert space for “distinguishable” systems, in which PI is not imposed. According to our interpretative principle regarding unitarily equivalent reps (2.4), we ought therefore to give the same physical interpretation to both joint Hilbert spaces. Since in the “distinguishable” case, we are happy to say that our systems are discerned, since they always possess different properties (their corresponding single-system Hilbert spaces share no state in common), we ought to be happy to say the same in the case of our so-called “indistinguishable” systems. Therefore, if the assembly’s state lies completely in an individuation block, the constituent systems are discernible.

What kind of discernibility do we have here? A modest tradition of taxonomizing grades of discernibility was inaugurated by Quine (1960), and has been continued by Saunders (2003b, 2006), Ketland (2011), Caulton & Butterfield (2012a), Ladyman, Pettigrew and Linnebo (2012); and was crucial to identifying the subtle kind of discernibility in the results of Muller and Saunders (2008), Muller and Seevinck (2009), Caulton (2013) and Huggett and Norton (2013). There is a general consensus that there are three main grades, here defined in model-theoretic terms. We fix some language \( L \). Then:

- **Absolute discernibility.** Two objects \( a \) and \( b \) are absolutely discernible in some model \( \mathfrak{M} \) iff there is some monadic formula \( \phi(x) \), expressible in \( L \) and not containing any singular terms, such that \( \mathfrak{M} \models \phi(a) \) and \( \mathfrak{M} \models \neg \phi(b) \).

- **Relative discernibility.** Two objects \( a \) and \( b \) are relatively discernible in some model \( \mathfrak{M} \) iff there is some dyadic formula \( \psi(x, y) \), expressible in \( L \) and not containing any singular terms, such that \( \mathfrak{M} \models \psi(a, b) \) and \( \mathfrak{M} \models \neg \psi(b, a) \).

- **Weak discernibility.** Two objects \( a \) and \( b \) are weakly discernible in some model \( \mathfrak{M} \) iff there is some dyadic formula \( \psi(x, y) \), expressible in \( L \) and not containing any singular terms, such that \( \mathfrak{M} \models \psi(a, b) \) and \( \mathfrak{M} \models \neg \psi(a, a) \).

I say that there are three grades, since it may be shown that absolute entails relative discernibility, and relative entails weak (Quine (1960, 230)). Therefore absolute discernibility is the strongest kind. And it is the kind exhibited by qualitatively individuated
systems, since these systems (like their “distinguishable” counterparts) enjoy the physical interpretation of having orthogonal single-particle states, and the occupation of a single-particle state is expressible with a monadic predicate.

In summary: if the assembly’s joint state lies in an individuation block, then its constituent systems are absolutely discernible. So our question now is, In what cases does the assembly’s joint state lie in some individuation block?

6.2 Absolutely discerning fermions, and “quantum counterpart theory”

Here we use the results of sections 5.2 and 5.3. In the discussion of fermions in section 5.3 we saw that any 2-fermion joint state can be ‘Slater decomposed’, and that individuation criteria may be chosen such that the joint state has support entirely within the individuation block so defined. It follows that fermions are absolutely discernible in all states.

It is worth emphasising that this result does not depend on the fermions in question possessing pure states; in fact the general form (107) of the 2-fermion joint state entails that the reduced density operators for the constituent fermions will typically be mixed. So I am not here appealing to the undeniable fact that fermionic Fock space quanta—which are always in pure states—are always absolutely discernible. In fact, this latter result follows as special case of our general result for fermions, in which the individuation criteria are all 1-dimensional.

Recall too that there is conventionality in the choice of individuation criteria which determine an individuation block that will capture the joint state. Given the generic ‘Slater decomposed’ form for the joint state,

\[ |\Psi_–\rangle = \sum_{i=1}^{\frac{d_j}{2} - 1} z_i \frac{1}{\sqrt{2}} (|\chi_{2i-1}\rangle \otimes |\chi_{2i}\rangle - |\chi_{2i}\rangle \otimes |\chi_{2i-1}\rangle) \]

\[ \equiv \sum_{i=1}^{\frac{d_j}{2} - 1} z_i |\chi_{2i-1}\rangle \wedge |\chi_{2i}\rangle, \]

(125)

(126)

(where we use for convenience the short-hand notation

\[ |\phi\rangle \wedge |\chi\rangle := \frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle - |\chi\rangle \otimes |\phi\rangle), \]

which, in analogy with differential geometry, we might call the wedge product of \( |\phi\rangle \) and \( |\chi\rangle \); I will say more about the wedge product in Section 7), there are \( 2^{(\frac{d_j}{2} - 1)} \) different choices for the pair of individuation criteria, all of which yield a suitable individuation block, as illustrated in Figure 5.

The physical interpretation of this is that there is no unique objective “trans-state identity” relation for qualitatively individuated systems. As I remarked in section 3.1,
\[ |\Psi_-\rangle = z_1|\chi_1\rangle \land |\chi_2\rangle + z_2|\chi_3\rangle \land |\chi_4\rangle + \ldots + z_{i\frac{1}{2}d}|\chi_{2i\frac{1}{2}d-1}\rangle \land |\chi_{2i\frac{1}{2}d}\rangle \]

\[ |\Psi_-\rangle = z_1|\chi_1\rangle \land |\chi_2\rangle + z_2|\chi_3\rangle \land |\chi_4\rangle + \ldots + z_{i\frac{1}{2}d}|\chi_{2i\frac{1}{2}d-1}\rangle \land |\chi_{2i\frac{1}{2}d}\rangle \]

\[ \vdots \]

Choices on each branch:

\[ 1 \times 2 \times \ldots \times 2 = 2^{(i\frac{1}{2}d-1)} \]

Figure 5: Choices of individuation criteria for an arbitrary 2-fermion joint state.

This suggests a quantum analogue of Lewis’s (1968) Counterpart Theory. In that theory, intended to provide an alternative to orthodox modal logic, there is no unique, objective “trans-world identity” relation between individuals in different possible worlds; the result being that many modal claims depend for their truth on a (usually contextually determined—or under-determined!) choice of “counterpart relations” between individuals from different possible worlds.

Here, the lack of any unique, objective “trans-state identity” relation affects not only modal claims but also non-modal claims. This is due to the possibility in quantum mechanics of superposing states to yield new states; so in particular, certain facts regarding a given GMW-entangled state hang on a choice of “counterpart relations” between the constituent systems in each of the non-GMW-entangled “branches” of which it is a superposition. These counterpart relations are determined by a choice of individuation criteria, and the dependence of certain facts on this choice is reflected in the dependence on the choice of individuation criteria of the effective joint state, as prescribed by (17).13

The situation for bosons is partly given by the above results for fermions, together with the unitary equivalence results we have made so much use of. Specifically: for any GMW-entangled bosonic joint state lying in an individuation block, we can expect there to be an alternative choice of individuation criteria, and therefore an alternative individuation block, which captures the state. Quantum counterpart theory goes for bosons as much as for fermions.

13Strictly speaking, an individuation criterion determines a counterpart relation that is more constrained than those permitted by Lewis. Lewis’s counterpart relations are deliberately designed so that they need not be equivalence relations; whereas the relation ‘satisfies the same individuation criterion E as’ is manifestly an equivalence relation.
In fact in the bosonic case it may be argued that the situation is closer to Lewis’s scheme: take for example the (GMW-entangled) state

$$|\Psi\rangle = \alpha|\phi\rangle \otimes |\phi\rangle + \beta \frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle + |\chi\rangle \otimes |\phi\rangle)$$ (128)

No individuation block captures this state, but it is sensible to extend our ideas of qualitative individuation and say that the two systems in $|\phi\rangle$ in the first “branch” (with amplitude $\alpha$) are both counterparts to the system in $|\phi\rangle$ in the second “branch” (with amplitude $\beta$). Here the counterpart relation between branches determined by the individuation criterion $|\phi\rangle\langle\phi|$ fails to pick out a unique system in every branch, just as Lewis’s counterpart relation is permitted to fail to pick out a unique individual in every world.

6.3 Discerning bosons: whither weak discernibility?

As usual, anything said of fermionic joint states also applies to bosonic joint states that lie in some individuation block. Therefore, both the absolute discernibility and the conventionality of “trans-state identity” carries over for bosons—so long as the joint state lies in an individuation block.

However, In section 5.2, we saw that there are bosonic states that lie in no individuation block: these states have non-zero amplitudes for multiply-occupied single-system states that cannot be removed under any change of basis in the single-system Hilbert space (in Figure 1, this corresponds to non-zero amplitudes for square lying along the diagonal). Our requirement that individuation criteria for distinct systems be orthogonal (one which the unitary equivalence result depends) means that these states can be captured by no individuation block.

Moreover, in the jargon of Ghirardi & Marinatto (2003), we may say of doubly-occupied bosonic states that both bosons have a complete set of properties, as determined by some 1-dimensional projector. Our unitary equivalence results do not apply here (since the joint state lies outside any individuation block), so there are no clues from there about how to interpret these states. But, given that the number operator for some single-system state $|\phi\rangle$ has eigenvalue 2 in such a joint state, it is natural to interpret this joint state as constituted by two systems, both in the state (represented by) $|\phi\rangle$.

Since the two bosons share the same state, we must conclude that they are not absolutely discernible. It appears too that they must not be relatively discernible. Might they yet be weakly discernible? The claim that bosons are weakly discernible has been defended recently (by Muller & Seevinck (2009), Caulton (2013), Huggett & Norton (2013)). However, these claims depend for their physical significance on the assumption that factor Hilbert space labels represent or denote constituent systems, i.e. they assume factorism.

Take for example Theorem 2 of Caulton (2013):
For each state $\rho$ of an assembly of two particles, the relation $R'(Q, x, y)$ [defined by]

$$R'(A, x, y) \iff \frac{1}{4} \text{Tr} \left[ \rho \left( A^{(x)} - A^{(y)} \right)^2 \right] \neq 0 \quad (129)$$

where $A^{(1)} := A \otimes 1$ and $A^{(2)} := 1 \otimes A$ discerns particles 1 and 2 weakly

...where $Q$ is the single-particle position operator, on the assumption of the Born rule.

The idea behind this theorem is that, even for product states with identical factors, if the assembly comprises more than 1 constituent, then this is indicated by anti-correlations in some single-system beable of which the constituent systems’ states are not eigenstates. These anti-correlations are taken to be measured by the (symmetric) beable $\Delta^2_A := \frac{1}{4} \left( A^{(1)} - A^{(2)} \right)^2$. (We then take advantage of the fact that no eigenstates of $Q$ exist in the single-system Hilbert space.)

However, despite that fact that $\frac{1}{4} \left( A^{(x)} - A^{(y)} \right)^2$, and therefore the relation $R'(A, x, y)$, is permutation-invariant for any $A, x, y$, the condition on the RHS of (129) may be interpreted as a relation between two particles only if $x$ and $y$ take particle names as substitution instances. In fact, they take factor Hilbert space labels as substitution instances (as indicated by the definitions of $A^{(x)}$), so it is only if factorism is right—which it is not—that the condition counts as a genuine relation between physical systems. Thus the weak discernibility claim fails.

However, the theorem undoubtedly has some physical interpretation. Can’t we still say of our two bosons in the joint state $|\phi\rangle \otimes |\phi\rangle$ that their states exhibit anti-correlations in some basis? Here we run into difficulty. Under anti-factorism, the only way to refer to constituent systems is via individuation criteria—in this case we say that there are two bosons in the state $|\phi\rangle$. Therefore the identity of the systems is inextricably tied to our choice of individuation criteria: if we change the criteria, then we change the subject. Therefore, to be able to talk about the sort of anti-correlations of which Caulton’s (2013) Theorem 2 takes advantage, we would have to shift our individuation criteria to that basis, but by doing so we would thereby be talking about a different collection of bosons. Our original question, whether the two bosons in state $|\phi\rangle$ are weakly discernible, remains unanswered.

This peculiar individuation-criterion-dependent nature of talk about constituent systems is unavoidable for an anti-factorist. As we shall see in section 7, this takes a particularly puzzling form in the case of fermions. But first, I will make a few comparative remarks about the qualitative individuation of classical point particles.

6.4 Comparisons with classical particle mechanics

At this point it may be illuminating to mark the similarities and differences between classical point-like particles and fermions and bosons. Just as, in quantum mechanics,
under PI we are led to consider the joint space of symmetric \((\mathfrak{h}_r^{(N)})\) or anti-symmetric \((\mathfrak{h}_\omega^{(N)})\) states, in classical mechanics, under PI, we are led to consider the reduced phase space (RPS) \(\Gamma_{\text{red}}\), which is the standard phase space \(\Gamma\)—a Cartesian product of \(N\) copies of the single particle phase space, \(\Gamma = \prod^N \gamma\)—quotiented by the symmetric group \(S_N\) (see Belot 2001, 66-70).

The classical version of PI states that for all quantities \(f\) with a physical interpretation, and all \(\pi \in S_N\),

\[
(f \circ P(\pi))(\xi) = f(\xi)
\]  

for all joint states \(\xi \in \Gamma\), where the symplectomorphism-valued function \(P\) is defined by

\[
P(\pi)(\langle q_1, p_1, \ldots, q_n, p_n \rangle) := \langle q_{\pi^{-1}(1)}, p_{\pi^{-1}(1)}, \ldots, q_{\pi^{-1}(n)}, p_{\pi^{-1}(n)} \rangle.
\]

Classical PI entails that all states in the same \(S_N\)-orbit give the same values for all physical quantities \(f\), and it is this that justifies quotienting \(\Gamma\) by \(S_N\).

Individuation blocks in the quantum case correspond to “off-diagonal” rectangular regions of the RPS; these regions have a Cartesian product structure, rather than the individuation blocks’ tensor product structure, and they behave just like phase spaces for “distinguishable” particles (i.e. when PI is not imposed).

More specifically, let \(E_i, i = 1, \ldots, N\) be \(N\) disjoint regions of the single-particle phase space, i.e. \(E_i \subseteq \gamma\), and \(E_i \cap E_j = \emptyset\) for all \(i \neq j\). Then the joint phase space \(\Gamma_D := \prod^N E_i\) represents physical states in which all \(N\) distinguishable particles necessarily have different positions or momenta. We may now define a symplectomorphism \(h : \Gamma_D \to R\) from this joint phase space onto a region \(R \subset \Gamma_{\text{red}}\):

\[
h(\xi) := \{(P(\pi))(\xi) \mid \pi \in S_N\}.
\]

\(h\) maps points of \(\Gamma_D\) to their orbits under \(S_N\); these orbits are the “points” of \(R\). \(h\) induces a “drag-along” map \(h^*\) on quantities on \(\Gamma_D\); any quantity for “distinguishable” particles \(f : \Gamma_D \to \mathbb{R}\) is sent to a permutation-invariant quantity \(h^*f\) for “indistinguishable” particles, according to the rule \((h^*f)(\xi) := (f \circ h^{-1})(\xi)\), for all \(\xi \in R\).

This symplectomorphism warrants us to afford the same physical interpretation to the states of \(R\) that we afford to the states of \(\Gamma_D\); in particular, that the constituent particles are absolutely discernible, since they all possess different positions or momenta. So long as the system point lies in \(R\), we can adjudicate matters of trans-state (and therefore also trans-temporal) identity, talk about the same particles having different positions and momenta in different states, etc. But we are mindful that the existence of different product phase spaces \(\Gamma'_D \subset \Gamma\) with different symplectomorphisms \(h'\) such that \(h' : \Gamma'_D \to R\) forces us to recognise a conventionality in these adjudications. (See Figure 6).

No single individuation region (the term we may coin for regions such as \(R\)) can capture the entire RPS \(\Gamma_{\text{red}}\), since \(\Gamma_{\text{red}}\) does not have a global Cartesian product structure. However, some individuation region may be found for almost all joint states: the only
states that cause trouble are those in which two particles have exactly the same position and momentum. (These states already cause trouble, since they lie on a boundary in the RPS, and therefore one cannot define tangent vectors for them.)

This far, the qualitative individuation of classical particles under PI is entirely analogous to that of their quantum counterparts. This is perhaps a surprise: for the quantum joint state, unlike the classical joint state, is not confined to a single point of the state space, but may rather be conceived as a wave that may spread throughout all of configuration space; yet, in the case of fermions and for most bosonic states, the state may still be confined to some individuation block.

These analogies should also bring reassurance to those wishing to conceptually reconcile rival approaches to imposing PI in quantum mechanics. This article has focussed exclusively on the procedure in which we first quantize and then impose PI; but one may instead proceed by first imposing PI on the classical joint phase space, and then quantizing. The result is that (for 3 spatial dimensions or more) one automatically yields states belonging to either the boson or fermion sectors of $\mathcal{F}_N^{(N)}$ (see Leinaas & Myrheim (1977), Morandi (1992, ch. 3)). We are therefore justified in thinking of the qualitative individuation present in this section 3 as the quantized version of qualitative individuation for classical particles on the RPS, as just outlined.

However, there are significant differences between the classical and quantum cases. I will discuss two here; they may both be considered a consequence of the possibility of quantum superpositions and incompatible beables:

1. **Disjunctive individuation.** I remarked at the end of section 3.3 above that one cannot qualitatively individuate by using multiple individuation blocks at once, since this induces a superselection rule between individuation blocks and one therefore
loses important information about the joint state. The same is not true in the classical case: that is, one may mosaic the entire RPS—excluding the “diagonal” points—with individuation regions, and thereby capture every joint state that does not lie along the “diagonal”. (I call this “disjunctive individuation”, since it has the logical form of a disjunction of conjuncts: e.g. ‘either particle 1’s state lies in \( E_1^{(1)} \) while particle 2’s state lies in \( E_2^{(1)} \) . . . , or particle 1’s state lies in \( E_1^{(2)} \) while particle 2’s state lies in \( E_2^{(2)} \) . . . , or . . . ’)

This also means that one may make the individuation regions as small as one likes without losing information. This, in effect, is what is going on when we opt for a relational means of discerning the two classical particles. For example, for any 2-particle state in which the 1-particle configuration space is \( \mathbb{R} \), the choice to label as particle “1” the particle that lies leftmost along the line (i.e. \( q_1 < q_2 \)) is equivalent to foliating the joint phase space with infinitely many individuation regions \( R(q) \) (one for each value of \( q \)) in which the corresponding \( \Gamma_D(q) := E_1(q) \times E_2(q) = \{ \langle q_1, p_1; q_2, p_2 \rangle \mid q_1 \leq q \& q_2 > q \} \).

MENTION THAT THIS MAKES THE PI VERSION OF CM EXPERIMENTALLY INDISTINGUISHABLE FROM THE STANDARD VERSION.

2. Incommensurable individuation criteria and composition. Another significantly novel aspect of qualitative individuation in quantum mechanics is that the same state may be captured by two different individuation blocks determined by mutually incompatible pairs of individuation criteria. For example, the non-GMW entangled fermion state

\[
\frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle - |\chi\rangle \otimes |\phi\rangle)
\]

may be captured by the 1-dimensional individuation block defined by the criteria

\[
E_{\alpha}^{(1)} := |\phi\rangle \langle \phi|; \quad E_{\beta}^{(1)} := |\chi\rangle \langle \chi|
\]

but this block is defined also by the incompatible criteria

\[
E_{\alpha}^{(2)} := |\phi'\rangle \langle \phi'|; \quad E_{\beta}^{(2)} := |\chi'\rangle \langle \chi'|
\]

where

\[
|\phi'\rangle := \alpha|\phi\rangle + \beta|\chi\rangle; \quad |\chi'\rangle := -\beta^*|\phi\rangle + \alpha^*|\chi\rangle
\]

and \( \alpha, \beta \neq 0 \) or 1.

This phenomenon, which arises only for fermions but not bosons, has serious consequences for the sense in which an assembly is composed of its constituent systems. I take the matter up in the following section.
7 A preferred basis problem

In this section I pick up a theme last mentioned at the ends of sections 3.6, 6.3 and 6.4, regarding the apparent individuation-criterion-dependent nature of talk about constituent systems under anti-factorism. The puzzling nature of this dependence affects both bosons and fermions, but it has a particularly vivid form for fermions, so I will give most attention to it. In section 7.1 I will introduce the general idea of composition and compare the classical and quantum cases. There I will argue that (GMW-) entanglement presents a problem for understanding the composition of “indistinguishable” (but not “distinguishable”) quantum systems. I then turn to a similar problem, which regards non-GMW-entangled joint states of fermions. In section 7.2, I will argue that a particularly apt way to represent a non-GMW-entangled joint state of $N$ fermions is as an $N$-dimensional subspace in the single-system Hilbert space $\mathcal{H}$. The negative, or at least puzzling, implications of this are discussed in section 7.3 and I outline there an overview of potential solutions. These solutions are then discussed in sections 7.4 to 7.8.

7.1 Composition, classical and quantum

Following Armstrong (1997), we may say that an object—a classical particle or an assembly of such particles; a quantum system or assembly of such systems—is a ‘thick particular’, that is, a certain state of affairs. That idea, inspired by the early Wittgenstein, is part of a detailed metaphysical system, but all it needs to mean here is that an object can be thought of as an instantiated, or realized, state or collection of properties. Consequently, objects may be aptly represented as states in some relevant state space.

There is not (or need not be) any confusion here between objects and possible states, which are also represented this way. An object, according to this idea, is simply a possible state that is instantiated or realized; it therefore inherits the properties associated with the state it realizes, and so can be represented in the state space just as the state is.

Classical particles and mereological structure. In the case of classical particles, we may represent a particle by a point in the single-particle phase space—the point corresponding to the state that the particle instantiates. Any joint state may then be represented, completely and without redundancy, by a cluster of points in the single-particle phase space. (In statistical mechanics, the single-particle phase space, used in this way, is often called “$\mu$-space”.) The compositional structure of the particles then inherits the structure governing these points and their clusters. I will call such a structure mereological.

NEED TO BE SPECIFIC ABOUT THE MEREOPLOGICAL THEORY:

The theory has $\subseteq$ as a partial order (reflexive, transitive and anti-symmetric). We then assume strong supplementation, general sum and product. (There is a subtlety here: do we assume unrestricted composition, or merely general sum? If the latter, then infinite domains need not be closed under fusion. I am dealing only with finite assemblies,
so I neglect this issue.) The result is known as classical extensional mereology, or CEP. (In fact, Top and Atomicity will also be true. If we assume unrestricted composition, then we have the general extensional mereology of Leśniewski 1916/1992 and Leonard & Goodman 1940.)

Assuming only finite domains, classical particle mechanics obeys atomistic extensional mereology AEM (i.e. the former’s models are models for the latter). AEM assumes P1-P3, P5 (strong supplementation) and P8 (atomicity). In fact it obeys atomistic general extensional mereology, AGEM, with complementation.

Fermions surely? disobey P6 (complementation)—consider a plane and a state-vector that it contains.

It is worth reviewing the formal properties of this mereological structure. We start with the mereological parthood relation (or just: the parthood relation) \( \sqsubseteq \), which relates parts to their wholes, just as points are related to point-clusters and point-clusters are related to larger point-clusters. This relation is reflexive (due to the possibility of improper parthood, i.e. identity), transitive and anti-symmetric, so it is a partial order.

With the parthood relation \( \sqsubseteq \) we may define two functions: fusion and common parthood. Given any two objects \( a \) and \( b \), their fusion \( a \sqcup b \) is defined as the minimal (with respect to \( \sqsubseteq \)) object having both \( a \) and \( b \) as parts. Any two objects have a fusion (the assumption of unrestricted composition). Given any two objects \( a \) and \( b \), the common part \( a \circ b \) is defined as the maximal (with respect to \( \sqsubseteq \)) object that is a part of both \( a \) and \( b \). There may be no such object, so \( \circ \) is a partial function. The resulting structure may be embedded into a Boolean lattice (with fusion as supremum and common parthood as infimum) that differs only by having an additional element, the null fusion \( * \), which is minimal with respect to \( \sqsubseteq \) (see e.g. Leonard & Goodman (1940, 46)). A key aspect of the quasi-Boolean nature of this mereological structure is that common parthood distributes over fusion: i.e. for all \( a, b, c \): \( a \circ (b \sqcup c) = (a \circ b) \sqcup (a \circ c) \). (Technical note: we need a semantics in which \( x = x \) is true even when \( x \) is empty, to cover the case in which \( a \) and \( b \sqcup c \) have no common part.)

“Distinguishable” quantum systems. In the case of “distinguishable” quantum systems and their unentangled joint states, we may say something similar, at least for unentangled states. This time, we may represent a constituent system with a ray in the single-system Hilbert space \( \mathcal{H} \)—the ray that the system instantiates. Unentangled joint states are uniquely determined by the single-system states of the constituents, so an unentangled joint state may be represented, completely and without redundancy, by a “bundle” of rays in \( \mathcal{H} \). Here again the compositional structure is mereological: the relation between rays and their bundles, and between bundles and larger bundles, is reflexive, transitive and anti-symmetric; and the “common rays” function distributes over bundle fusion.

The similarity with the classical case can be made even clearer if we represent single-system quantum states not with rays in \( \mathcal{H} \) but, as comes to the same thing, with points in the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \). (This space is a symplectic manifold, and therefore a
phase space. This insight is just the beginning of the fascinating “geometrical” approach to quantum mechanics inaugurated by Kibble (1979), which contains rich analogies with classical mechanics. Ashtekar & Schilling (1999) is also an excellent introduction.) Under this convention, any unentangled joint state, previously represented by a bundle of rays in $\mathcal{H}$, is represented, like in the classical case, by a cluster of points in the single-system “phase space” $\mathcal{P}(\mathcal{H})$, and parthood has its usual representation as the relation between points and point-clusters.

Of course, entanglement presents an obstacle to a complete analogy between the classical and quantum cases. Since entangled states are not uniquely determined by the reduced states of the constituent systems, they are not well represented in the single-system spaces $\mathcal{H}$ or $\mathcal{P}(\mathcal{H})$. In the case of “distinguishable” systems, the correct physical interpretation of this is straightforward. In short, entanglement between two systems is construed as an irreducible relation holding between them.

This relation can be represented only in the joint Hilbert space. This should be expected: any relation between a collection of objects can equally well be represented as a property (a state) of the fusion of those objects. (We normally take advantage of this equivalence in the opposite direction: e.g. the shape of a triangle, construed as a fusion of three vertices, is determined by the distance relations holding between those vertices.) In particular, there is no call for holism-inspired overreactions, along the lines that the “only real object” is the entire quantum assembly. Rather, if we insist on representing all physical quantities—including irreducible relations—as states of some system, then we oughtn’t be surprised if we have to refer to the largest system to do so.

“Indistinguishable” quantum systems. The deflating comments above may be reassuring to those who seek a homely physical picture of quantum systems. Unfortunately, this story needs substantial revision in the case of “indistinguishable” systems. We saw above (section 6.2) that anti-factorism entails that we must give up any notion of a unique, objective trans-branch identity relation for “indistinguishable” systems. If that is so, then (GMW-) entanglement for such systems cannot be construed, as with “distinguishable” systems, as an irreducible relation holding between constituent systems.

The relation in question is constituted by the existence of several non-GMW-entangled branches in the joint state, and so requires the systems to be non-arbitrarily identifiable between the branches. The individuation criteria I have advocated supply at best a family of “counterpart relations” between systems across branches: but a change in individuation criteria means a change in counterpart relations, and therefore a change in the relations that purportedly constitute the entanglement. But if the relations are real (as entanglement undoubtedly is), then they cannot be subject to anything as arbitrary as the individuation criteria we choose to impose.

We must proceed under the undeniable absence of any objective trans-branch identity relations. But how are GMW-entangled states to be understood then? One suggestion is that they be understood as co-existing non-GMW-entangled branches, related to each other by their relative amplitudes. On this conception a GMW-entangled state is like an
archipelago, with each non-GMW-entangled branch for which it has a non-zero amplitude as an individual island. The composition of GMW-entangled states can then be seen, one hopes, as a two-stage process: first, each non-GMW-entangled branch is composed of its constituent systems in the manner familiar to the unentangled states of “distinguishable” systems and classical particles; and second, the GMW-entangled state is nothing but a particular family of relative amplitudes holding between these branches.

But there are two problems with this proposal (in fact we shall see that they are instances of the same general problem). First of all, it seems that the composition of non-GMW-entangled fermionic states cannot be construed along the same lines as unentangled joint states of “distinguishable” systems and classical particles; this is the subject of the following sections. Secondly, putting aside potential problems in understanding the composition of the non-GMW-entangled branches, the picture of GMW-entangled states being constituted by branches related to each other by certain determinations of relative amplitude is complicated by the fact that the same GMW-entangled state may be broken down into non-GMW-entangled branches in a variety of ways.

For example, the state

$$|\Psi\rangle = a|\phi\rangle \otimes |\phi\rangle + b \frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle + |\chi\rangle \otimes |\phi\rangle) + c|\chi\rangle \otimes |\chi\rangle$$

(137)

(where $\langle \phi | \chi \rangle = 0$) may also be written

$$|\Psi\rangle = a'|\phi'\rangle \otimes |\phi'\rangle + b'| \frac{1}{\sqrt{2}}(|\phi'\rangle \otimes |\chi'\rangle + |\chi'\rangle \otimes |\phi'\rangle) + c'|\chi'\rangle \otimes |\chi'\rangle$$

(138)

where

$$|\phi'\rangle = \alpha|\phi\rangle + \beta|\chi\rangle; \quad |\chi'\rangle = -\beta^*|\phi\rangle + \alpha^*|\chi\rangle;$$

(139)

and

$$a' = (\alpha^*)^2 a + \sqrt{2}\alpha^* \beta b + (\beta^*)^2 c;$$

$$b' = -\sqrt{2}\alpha^* \beta a + (|\alpha|^2 - |\beta|^2)b + \sqrt{2}\alpha \beta^* c;$$

$$c' = \beta^2 a - \sqrt{2}\alpha \beta b + \alpha^2 c.$$

(140)

for any $\alpha, \beta$ such that $|\alpha|^2 + |\beta|^2 = 1$. The following two physical interpretations of this state are then suggested:

1. The state is the superposition of three non-GMW-entangled branches. On the first branch there are two systems in $|\phi\rangle$; on the second branch there is one system in $|\phi\rangle$ and one in $|\chi\rangle$; on the third there are two systems in $|\chi\rangle$. The amplitude of the second branch relative to the first is $\frac{b}{a}$, and the relative amplitude of the third branch relative to the first is $\frac{c}{a}$.

2. The state is the superposition of three non-GMW-entangled branches. On the first branch there are two systems in $|\phi'\rangle$; on the second branch there is one system in $|\phi'\rangle$ and one in $|\chi'\rangle$; on the third there are two systems in $|\chi'\rangle$. The amplitude of the second branch relative to the first is $\frac{b'}{a'}$, and the relative amplitude of the third branch relative to the first is $\frac{c'}{a'}$. 62
In fact there are *continuum-many* possible interpretations: one for each choice of \(\alpha, \beta\). Which is correct? Surely we ought to construe all of them as correct. But how? Each talks of different systems in different states: the same assembly has rival decompositions into constituent systems!

The phenomenon that an object has many decompositions into parts is familiar in mereology. For example: a collection of four socks has six (not two!) different decompositions into two non-overlapping pairs. However, in the case of mereology, the plurality of decompositions relies on the parts not being elementary: the four socks taken as a whole has only one decomposition into four socks taken individually. In the above case, the state \(|\Psi\rangle\) has rival decompositions into what we would ordinarily hope to think of as elementary systems.

It seems, then, that we are forced to conclude that the systems in question are not, after all, elementary. But this prompts the question, what then are the elementary parts? The are not represented in the formalism, and we have no other reason to believe they exist, except for our procrustean ambition to construe the construction of joint quantum states as mereological.

The problem here is similar, but not entirely analogous, to that affecting the interpretation of non-GMW-entangled fermionic joint states. The common theme is that a tension exists between the familiar mereological structure of composition and structure dictated by the mathematical formalism. In the case just presented, the tension is between mereological fusion and the vector addition associated with quantum superposition. That is, the second stage of “composition” for quantum joint states—in which non-GMW-entangled branches are superposed to form a GMW-entangled state—has a vector space structure rather than a mereological structure. (As we shall see in the following sections, in the case of non-GMW-entangled fermionic states, the tension is between the mereological parthood relation and the subspace relation.)

The tension between mereological fusion and vector addition has not much been addressed, despite the ubiquity of vector spaces in both classical and quantum mechanics. There are two notable exceptions. Although not phrased in the same way, Cartwright (1983, 59-62) expresses scepticism that we may think of the “composition” of Newtonian forces (by vector addition) as a genuine form of composition, in which we may attribute a separate physical reality to the resulting parts. (As she writes (p. 59), “Nature does not ‘add’ forces. For the ‘component’ forces are not there, in any but a metaphorical sense, to be added”.) Any Newtonian force, being a vector is 3-space, has many decompositions into orthogonal (even non-orthogonal!) “component” forces, and we have no good reason to treat any one decomposition as any more natural than any other.

Saunders (2010) addresses quantum superposition in particular. However, Saunders’ use of mereology to understand quantum superposition assumes a preferred basis, given by decoherence. However, the appeal to a preferred basis convincingly resolves the tension between mereological composition and vector addition only in those cases—such as decoherence—in which it is physically reasonable to suppose that a preferred basis
has been selected. In our case above, no such basis is naturally privileged, so any such choice is arbitrary and *ad hoc*.

Until we have some way of understanding superposition as a genuine form of composition, our understanding of quantum assemblies as being “made out of” constituent systems will remain unclear—at least for the anti-factorist. It may be tempting to give up the ghost altogether and embrace holism: I return to this suggestion below.

7.2 Fermionic states as subspaces of the single-system Hilbert space

I will now focus exclusively on the first problem mentioned above: that of understanding the composition of non-GMW-entangled joint states of fermions. First we need to establish the mathematical structure of fermionic composition. To this end, in this section I propose a way of aptly representing non-GMW-entangled fermionic joint states in the single-system Hilbert space.

First, let us consider again the joint states of “distinguishable” systems. If any such joint state is unentangled, then it is uniquely determined by the states of its constituents. It follows that the space of unentangled joint states for an assembly of “distinguishable” systems has a Cartesian product structure with respect to the states of its constituent systems. That is, there is a natural bijection $f^{(N)} : \mathcal{H} \times \ldots \times \mathcal{H} \rightarrow \mathcal{H} \otimes \ldots \otimes \mathcal{H}$ from any $N$-tuple $(|\phi(1)\rangle, \ldots, |\phi(N)\rangle)$ of single-system states to an unentangled joint state $|\phi(1)\rangle \otimes \ldots \otimes |\phi(N)\rangle$.

As we have already seen, it is heterodox, but harmless, to represent unentangled joint states in the single-system Hilbert space $\mathcal{H}$, by specifying $N$ state-vectors or rays, and assigning each one a number corresponding to the constituent system in question. In this case one could consider $\mathcal{H}$ as playing a role analogous to $\mu$-space is classical statistical mechanics, in which a joint state of $N$ particles is represented in a single-particle phase space by a cluster of $N$ points. We also saw that the advantages of this heterodox means of representation are limited, since entangled states cannot be represented. But insights are gained when we consider the equivalent situation for fermionic joint states.

I have argued (in section 5.2) that non-GMW-entangled states are the fermionic or bosonic equivalent to unentangled states for “distinguishable” systems, since for these states too, the joint state is uniquely determined by the states of the constituent systems (or, equivalently: by a specification of occupation numbers for the single-system states). So take an arbitrary non-GMW-entangled state of $N$ fermions,

$$|\Psi_-\rangle = |\phi(1)\rangle \wedge |\phi(2)\rangle \wedge \ldots \wedge |\phi(N)\rangle,$$

(141)

where $\phi : \{1, \ldots, N\} \rightarrow \{1, \ldots, d\}$ is an injective function, $\{|1\rangle, \ldots, |d\rangle\}$ is an orthonormal basis for $\mathcal{H}$, and I use the “wedge product” notation introduced in section 6.2. (Ladyman *et al* (2013, 219-20) also make use of this notation, but do not pursue its unusual consequences for the nature of fermionic $N$-system states.) Similar to the “distinguishable” case, we can uniquely specify $|\Psi_-\rangle$ by specifying $N$ state-vectors (now they must all be mutually orthogonal) in $\mathcal{H}$. 64
One difference, when compared to the “distinguishable” case, is that we need not specify which of the $N$ constituent systems possesses which of $N$ states. Or better: there is no longer any fact of the matter which system occupies which state, since the systems may be individuated only qualitatively: that is, by their states. We might represent this fact by declining to assign labels to the $N$ state-vectors. The result is a specification of $N$ state-vectors that can be taken in any order.

We would expect something similar in classical mechanics: once we accept that particles may be individuated only qualitatively, we no longer distinguish between clusters of points in $\mu$-space that differ only as to how the points are labelled. However, in the case of fermions, a much more dramatic under-determination besets the specification of the non-GMW-entangled joint state. The anti-symmetry of such states entails that they may be decomposed into many different rival families of constituent states.

The most celebrated example of this is the spherically symmetric spin-singlet state

$$|\psi_-\rangle := |\uparrow\rangle \wedge |\downarrow\rangle = |\theta, \phi\rangle \wedge |\theta, \phi + \pi \pmod{2\pi}\rangle,$$

where $|\theta, \phi\rangle$ is the positive eigenstate of the operator $u^i_{\theta, \phi}\sigma_i$, and the normalized spatial vector $u^i_{\theta, \pi} := (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta)$ is defined by a point on the unit sphere with a latitude $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and longitude $0 \leq \phi < 2\pi$ (so e.g. $|\uparrow\rangle \equiv |\frac{\pi}{2}, \phi\rangle$, for any $\phi$, and $|\rightarrow\rangle \equiv |0, 0\rangle$). But this arbitrariness is a property of any non-GMW-entangled 2-fermion state.

Since this arbitrariness involves fixing an orthonormal basis in a copy of $\mathbb{C}^2$, it may be parameterized by $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$, which may be identified with the Bloch sphere determined associated with that copy of $\mathbb{C}^2$. In fact, each orthonormal basis is represented by a pair of antipodal points on the Bloch sphere, since we can permute the two basis vectors without change. Thus a choice of basis corresponds to the choice of a single point in $\mathbb{C}P^1/S_2$, which is homeomorphic to the projective plane.

A vivid visual metaphor of this basis freedom is provided by the fact that a pair of antipodal points on the Bloch sphere specifies a line through its centre, and a plane orthogonal to that line that divides the sphere into two hemispheres. Thus the arbitrariness in the 2 single-system states into which a non-GMW-entangled 2-fermion joint state may be decomposed corresponds to the continuum-many ways one may divide a sphere into two hemispheres; cf. Figure 7.

This arbitrariness is only exacerbated by the addition of more fermions. To see this, it is enough to notice that total anti-symmetrization is constituted by all possible pairwise anti-symmetrizations. For example, in the non-GMW-entangled 3-fermion state
Figure 7: Two ways to halve the Bloch sphere corresponding to the singlet state. Each halving corresponds to a pair of orthogonal single-system states.

\[ |\Phi_\perp\rangle \], where we define \( s(i) := (i + 1) \mod 3 \) (so \( s^3(i) \equiv i \)),

\[
|\Phi_\perp\rangle = \frac{1}{\sqrt{6}} \begin{vmatrix} |\alpha\rangle_1 & |\beta\rangle_1 & |\gamma\rangle_1 \\ |\alpha\rangle_2 & |\beta\rangle_2 & |\gamma\rangle_2 \\ |\alpha\rangle_3 & |\beta\rangle_3 & |\gamma\rangle_3 \end{vmatrix}
\]

\[
= \frac{1}{\sqrt{6}} \sum_{i=1}^{3} |\alpha\rangle_i \otimes \left( |\beta\rangle_{s(i)} \otimes |\gamma\rangle_{s^2(i)} - |\gamma\rangle_{s(i)} \otimes |\beta\rangle_{s^2(i)} \right)
\]

\[
= \frac{1}{\sqrt{6}} \sum_{i=1}^{3} |\beta\rangle_i \otimes \left( |\gamma\rangle_{s(i)} \otimes |\alpha\rangle_{s^2(i)} - |\alpha\rangle_{s(i)} \otimes |\gamma\rangle_{s^2(i)} \right)
\]

\[
= \frac{1}{\sqrt{6}} \sum_{i=1}^{3} |\gamma\rangle_i \otimes \left( |\alpha\rangle_{s(i)} \otimes |\beta\rangle_{s^2(i)} - |\beta\rangle_{s(i)} \otimes |\alpha\rangle_{s^2(i)} \right)
\]

(143)

(where labels have been added simply to make the ordering in the tensor product clear), all of the bracketed two-fermion states are subject to the usual basis arbitrariness, each in a different 2-dimensional subspace of \( \mathcal{H} \).

Generally, any non-GMW-entangled state of \( N \) fermions is subject to a basis arbitrariness that is parameterized by the manifold

\[
\left( \mathbb{CP}^{N-1} \times \mathbb{CP}^{N-2} \times \cdots \times \mathbb{CP}^1 \right) / S_N
\]

(144)

(where we quotient by the natural group action of \( S_N \), since a permutation of basis vectors does not change the basis). This manifold has \((N - 1)!\) complex, and so \((N - 1)!2^{N-1}\) real, dimensions.

The points in this manifold correspond to a choice of \( N \) orthonormal vectors in an \( N \)-complex-dimensional space, the \( N \)-dimensional subspace of \( \mathcal{H} \) that is spanned by the
Figure 8: Along the top: Representations of: (a) a single-system state $|\phi\rangle$; (b) an unentangled joint state of 2 “distinguishable” systems $|\phi\rangle \otimes |\chi\rangle$; (c) a non-GMW-entangled state of 2 fermions $|\phi\rangle \wedge |\chi\rangle$; all represented in the single-system Hilbert space. Along the bottom: Corresponding representations of these states in the Bloch sphere defined by the two single-system states $|\phi\rangle$ and $|\chi\rangle$. The fermionic joint state spans the entire Bloch sphere.

corresponding single-system states. Therefore a natural idea suggests itself: why not just represent the $N$-fermion joint state with the entire $N$-dimensional subspace? This way, the failure of the joint state to determine uniquely a family of $N$ constituents is represented by the failure of an $N$-dimensional space to uniquely determine a family of $N$ orthogonal rays whose span yields that space.

In this way, what I called a “wedge product” really deserves its name. In differential geometry the wedge product also takes us from lower- to higher-dimensional geometrical objects: the wedge product of an $m$-form (which smoothly assigns an $m$-dimensional “volume” element to every point of space), with an $n$-form yields an $(m + n)$-form. Here, the “wedge product” of a non-GMW-entangled $M$-system joint state with a non-GMW-entangled $N$-system joint state yields a non-GMW-entangled $(M + N)$-system joint state. The case for $M + N = 2$ is shown in Figure 8.

I will note here one heuristic advantage of this representational device, apart from
avoiding basis arbitrariness. It helps explain the result above (section 5.3) that GMW-entanglement for 2 fermions is not possible until the dimension of the single-system Hilbert space $\mathcal{H}$ is 4 or more. Non-GMW-entangled joint states of 2 fermions correspond to 2-dimensional subspaces of $\mathcal{H}$, so if $\dim(\mathcal{H}) = 3$, then these joint states have codimension 1. Consequently, vector addition can be defined for them; i.e., any two such states may be superposed to yield another. Therefore, with arbitrary superpositions of non-GMW-entangled joint states, we never get anything other than another non-GMW-entangled state (see Figure 9). This reasoning can be generalized to single-system Hilbert spaces of higher dimension: in general, we may say that GMW-entanglement does not arise for an assembly of $N$ fermions unless $\dim(\mathcal{H}) \geq N + 2$.

### 7.3 A puzzle regarding fermionic composition

I have just argued that we may represent a non-GMW-entangled $N$-fermion joint state with an $N$-dimensional subspace of $\mathcal{H}$. Following the ideas of section 7.1, we may represent a particular assembly of $N$-fermions the same way. We may now read off the structure of composition for such assemblies and their constituents from the mathematical structure of their most apt representations as subspaces.

**HERE’S THE THING:** Fermionic composition is not fusion. the fermionic “fusion” of two objects is the minimal object having those two objects as parts. The fusion is the object such that any object $x$ overlaps it iff $x$ overlaps either object (see Casati and Varsi, SEP). These definitions coincide in the classical case but generally the former is weaker than the latter; i.e. the former allows more into the fusion than the latter; and only the latter produces a sum function that distributes with product. E.g. a 2-fermion state overlaps 1-fermion states that neither of the two 1-fermion states of which it is a fusion overlap. Thus, for fermions the usual sum axiom (M6) is false. The product axiom is satisfied, so I will continue to use $\circ$ to denote it.
The result is that the “fermionic parthood relation”—the relation a constituent fermion has to the assemblies of which it is a constituent—is not mereological. Mereological product, or common parthood, distributes over fusions: the common part of $a$ and the fusion of $b$ and $c$ is the fusion of the common part of $a$ and $b$ and the common part of $a$ and $c$. Meanwhile, fermionic “parthood” is represented by the subspace relation, so it is associated with the structure of a non-distributive orthomodular lattice.

The “fusion” of $a$ and $b$ is represented (like disjunction in quantum logic) by the linear span of $a$ and $b$, and “common parthood” is represented (like quantum conjunction) by intersection. So, for example, the “common part” (represented by intersection) of an assembly in the joint state $|↑⟩∧|↓⟩ ≡ |→⟩ ∧ |←⟩$ and one in the joint state $|→⟩ ∧ |φ⟩$ (where $⟨↑|φ⟩ = ⟨↓|φ⟩ = 0$) is a fermion in the state $|→⟩$, while neither the fermion in the state $|↑⟩$ nor the one in the state $|↓⟩$ has a “common part” with the assembly in the joint state $|→⟩ ∧ |φ⟩$; see Figure 10.

Another peculiar feature of fermionic “composition” is that the counterpart to mereological subtraction ($a − b :=$ the maximal part of $a$ that does share any parts with $b$) is relative orthocomplementation $⊖$, where $a ⊖ b :=$ the maximal subspace of $a$ that is orthogonal to $b$. This means that the “fusion” of all unoccupied single-system states is a subspace of $H$, and so has all the mathematical properties of a non-GMW-entangled joint state of a fermionic assembly. We are already familiar with this fact from solid state physics and Dirac’s theory of the electron: a fermion “hole” may be treated as a fermion in its own right.

Despite the discrepancies between classical and fermionic composition, it is worth pointing out that both classical particles and fermions compose in a way that reflects the structure of propositions in their respective theories. Classical composition shares the same Boolean structure as the classical-logical truth-functional operations on propositions regarding a single particle; fermionic composition shares the same structure as the quantal-logical operations on propositions regarding a single quantum system.

The discrepancy between the structures of fermionic “composition” and mereological composition present something of an interpretative puzzle. For naively we want to think of quantum systems composing their assemblies in the usual sense. It is worth emphasising here three points, regarding the nature and depth of the puzzle.

- **Nothing to do with entanglement.** First, the weirdness issues not from entangled states, but from non-entangled states: according to our anti-factorist interpretation, the puzzling joint states count as non-GMW-entangled. Of course, it follows that, since GMW-entangled states are superpositions of non-GMW-entangled states, the puzzle percolates upwards to these too.

(In fact, as we have seen, matters are worse for GMW-entangled states. For it is under-determined, at least for some states, which non-GM-entangled “branches” we should should say superpose to yield a given GM-entangled state. And this
Figure 10: Fermionic “composition” has the structure of a non-distributive orthomodular lattice. Here is a sublattice generated from 5 single-system states (bottom row), 4 of which are coplanar. The middle row shows the 5 non-GMW-entangled 2-fermion joint states which are possible to generate from them, and the top row shows the only non-GMW-entangled 3-fermion joint state possible to generate from them (there are no possible joint states of more than 3 fermions, given just these 5 single-system states). “Common part-hood” $\circ$ does not distribute over “fusion” $\wedge$, as seen in the example:

$|\rightarrow\rangle = (|\uparrow\rangle \wedge |\downarrow\rangle) \circ (|\rightarrow\rangle \wedge |\phi\rangle)$ $\neq$ $[|\uparrow\rangle \circ (|\rightarrow\rangle \wedge |\phi\rangle)] \wedge [|\downarrow\rangle \circ (|\rightarrow\rangle \wedge |\phi\rangle)] = \ast$, where $\ast$ denotes the null fusion.
applies even for bosonic assemblies.)

- **Not the typical failure of supervenience.** Second, the weirdness is not, as in entanglement, a result of a failure of supervenience of the joint state on the states of its constituents. On the contrary: the constituent fermions’ states determine the joint state. The problem is that very many different constituent fermion states determine the same joint state. Thus we may say that the failure of supervenience runs in the opposite direction: the joint state fails to determine the states of its constituents, or better, fails to determine the very constituents themselves.

- **Bad for Fock space too.** Finally, the puzzle blights not only the anti-factorist interpretation of quantum mechanics that I am here propounding; it blights also any particle-like interpretation of Fock space in QFT, due to the affinities between these two interpretations, as discussed in section 4.4. “No particle” results are familiar in QFT, but these focus either on the absence of a Fock space representation in the presence of interactions (see Fraser (2008)); the phenomenon of rival but incommensurable Fock space representations (see Clifton & Halvorson (2001)); or the absence of spatially local number operators in relativistic QFT (see Halvorson (2001) and Halvorson & Clifton (2002)). The puzzle here is more basic, in the sense that it pertains to any Fock space representation and does not appeal to the failure of localization.

I am pessimistic that this puzzle can be solved. Nevertheless, in this section, I will outline what I believe to be the best attempts to overcome it. By way of introduction, I list the five attempted solutions here. They may be categorised into two groups: the responses which attempt to overcome basis arbitrariness by finding, for each state, a uniquely privileged basis; and the responses which attempt to overcome the arbitrariness by somehow accommodating all bases at once, without privileging any one over the other. The first two responses fall under the former category; the second three responses fall under the latter. As I will argue in the following sections, response 1 is hopelessly ad hoc; 2 is incomplete; 4 is unbelievable; and 3 and 5—the seemingly most promising responses—lead to out-and-out contradiction.

1. **One size fits all.** There is a uniquely natural single-system basis for each state of the assembly. It is the same basis for every state. That is, there is a categorically privileged single-system basis.

   More technically, the strategy here is to restore a mereological (i.e. quasi-Boolean) structure to the structure of subspaces by removing all atoms (rays) save one orthonormal basis and all corresponding higher-dimensional spaces.

2. **Complicate.** In realistic cases, there is more than one degree of freedom under consideration. These extra degrees of freedom provide the extra structure needed to determine a uniquely natural single-system basis, for each state.

   In terms of the compositional structure, this strategy is a special case of One size fits all, in which the preferred basis is privileged by the physics.
3. **Coalesce.** All of the rival single-system bases may be reconciled by reifying all of the corresponding qualitatively individuated systems. But in each non-GM-entangled joint state, constituent systems associated with one single-particle basis are all identical to some constituent system associated with any other single-system basis.

This strategy is basically tantamount to positing “hidden variables”

...  

4. **Multiply.** All of the rival single-system bases may be reconciled by reifying all of the corresponding qualitatively individuated system. The systems in all bases are all distinct one from another.

This strategy is tantamount to reinterpreting the parthood relation. Rather than being represented by the subspace relation, under this proposal it is represented by the subset relation. This means, for example, that a “2-fermion” state (represented by a 2-dimensional subspace of \( \mathcal{H} \)) is taken actually to comprise continuum-many systems (each represented by a ray). Under this re-interpretation, quasi-Boolean structure is restored.

5. **Overlap.** All of the rival single-system bases may be reconciled by reifying all of the corresponding qualitatively individuated systems. But constituent systems associated with different bases are not wholly distinct. In fact systems in different bases overlap in such a way that for each non-GM-entangled state, the sum of all constituent systems in one single-particle basis are jointly identical to the sum of all constituent systems in any other single-particle basis.

The strategy here is to restore a quasi-Boolean structure by adding sufficiently many elements in the subspace structure. Thus non-orthogonal rays overlap by having a common part. Common parthood is no longer represented by intersection.

...  

7.4 The ‘One size fits all’ response

The One size fits all response is undeniably simple; but its simplicity issues from its rigidity, which is also the source of its drawbacks. It is clear that One size fits all easily and satisfactorily solves the basis ambiguity problem for non-GMW-entangled fermions, whenever one of the rival decompositions belongs to the categorically privileged basis. But if none of the rival decompositions belong to this basis, One size fits all comes into conflict with the result of section ??, that any non-GMW-entangled joint state corresponds to an assembly in which all the constituent systems possess pure states.

Furthermore, the One size fits all response is ad hoc: there is no reason to countenance a categorically privileged single-particle basis, except that it solves the basis ambiguity problem. And, of course, there is no uniquely natural suggestion for what
the privileged basis would be. (The two proposals that are perhaps the most intuitive, namely position and momentum, create particular trouble, since strictly speaking eigenstates for position or momentum do not exist in the single-system Hilbert space.) Therefore we turn to the next proposal.

7.5 The ‘Complicate’ response

The next proposal seeks to single out a preferred single-particle basis in a way that pays better attention to the physics. It takes advantage of the fact that, in realistic scenarios, particles have more than just one degree of freedom. Single-system Hilbert spaces for most existing particles include an internal spin space, and possibly other degrees of freedom, such as flavour and colour. To simplify the discussion, let us ignore these other degrees of freedom and consider only location and spin.

The utility of the spin degree of freedom in solving the basis arbitrariness problem lies in the extra structure it provides in breaking the under-determination of single-system bases. For, just as two “distinguishable” systems may be entangled, and two “indistinguishable” systems may be GMW-entangled, so the individual degrees of freedom associated with a single system may be entangled. Entanglement between the degrees of freedom of a single system is like entanglement between distinct “distinguishable” systems, in that the state is entangled if it is non-separable.

The proponent of Complicate stipulates that constituent systems only possess pure states when distinct degrees of freedom are not entangled; i.e. they may attributed both a pure spatial state and a pure spin state. The advantage of this proposal over One size fits all is that it uses the physical phenomenon of entanglement to determine a uniquely preferred single-system basis; so it could not be accused of being ad hoc.

Apart from that, Complicate shares two important features with One size fits all. First, it breaks the under-determination of bases when one of the rival bases yield systems whose states are not entangled in their degrees of freedom. For example, in the state

\[
\frac{1}{\sqrt{2}}(|L, \uparrow\rangle \otimes |R, \downarrow\rangle - |R, \downarrow\rangle \otimes |L, \uparrow\rangle),
\]

(145)

the single-system basis \{\{L, \uparrow\}, |R, \downarrow\}\} is uniquely preferred over its rivals, such as \{\{L, \uparrow\}\} + \frac{1}{\sqrt{2}}(|L, \uparrow\rangle - |R, \downarrow\rangle).\]

Second, however, the requirement that the single-system basis be non-entangled also conflicts with the principle that non-GMW-entangled states correspond to the constituent systems being ubiquitous and unique and occupying pure states. Define an uncomplicated system as a system whose state is non-entangled in its separate degrees of freedom. Then there are non-GMW-entangled states whose constituent ubiquitous and unique systems must be complicated.

\[14\text{Ghirardi et al (2002, p. 86) say of these rivals that they are ‘of no practical interest’. But I am trying to solve an ontological, not a practical, problem.}\]
For example, the single-system states
\[ |\phi\rangle := \alpha |L, \uparrow\rangle + \beta |R, \downarrow\rangle; \quad |\chi\rangle := \beta^* |L, \uparrow\rangle - \alpha^* |R, \downarrow\rangle \]
(146)

exhibit entanglement between the spatial and spin degrees of freedom (so long as \( \alpha, \beta \neq 0 \) or 1; so the systems occupying them count as complicated. Now consider the non-GMW-entangled (bosonic) state
\[ |\psi\rangle := \frac{1}{\sqrt{2}} (|\phi\rangle \otimes |\chi\rangle + |\chi\rangle \otimes |\phi\rangle) \]
\[ \equiv \sqrt{2} \alpha \beta^* |L, \uparrow\rangle \otimes |L, \uparrow\rangle - \sqrt{2} \alpha^* \beta |R, \downarrow\rangle \otimes |R, \downarrow\rangle \]
\[ + \left( |\beta|^2 - |\alpha|^2 \right) \frac{1}{\sqrt{2}} (|L, \uparrow\rangle \otimes |R, \downarrow\rangle + |R, \downarrow\rangle \otimes |L, \uparrow\rangle) \].
(148)

This state is a superposition of non-GM-entangled states in which the constituent ubiquitous and unique systems are uncomplicated.

Another problem for Complicate is that it will not work for every joint state, since joint states exist which continue to suffer a basis arbitrariness even when entanglement between degrees of freedom is taken into account. One such state is the ground state for the two electrons in a Helium atom:
\[ |\phi_{1s}\rangle_1 \otimes |\phi_{1s}\rangle_2 \otimes (|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2) \]
(149)
in which, for each qualitatively individuated electron, the states for each degree of freedom factorize. Here the demand that the constituent systems be uncomplicated helps not one bit in narrowing down the options.\(^{15}\) Therefore we must look elsewhere for a solution.

### 7.6 The ‘Coalesce’ response

An alternative strategy in solving the basis arbitrariness problem is to somehow accommodate all of the single-particle bases in which the assembly’s state may be expressed. The first response that adopts this strategy is Coalesce, which stipulates that constituent systems from one basis are identical to constituent systems from rival bases.

For example, for the singlet state
\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) = \frac{1}{\sqrt{2}} (|\rightarrow\rangle \otimes |\leftarrow\rangle - |\leftarrow\rangle \otimes |\rightarrow\rangle) \]
(150)
the proponent of Coalesce claims that the spin-up system (i.e. the system qualitatively individuated by the single-system state of being spin-up) is identical to either the spin-left system or the spin-right system, and that spin-down system is identical to whichever of these is not identical to the spin-up system.\(^{15}\)

\(^{15}\)This attempt to avoid the basis arbitrariness problem—and the example of the Helium ground state as a counterexample—has also been discussed by Thomas Bigaj.
This may be generalised for any number $N$ of systems, and also for bosons. Thus, according to the proponent of *Coalesce*, a non-GMW-entangled assembly is composed of $N$ systems each of which is qualitatively individuated by continuum-many single-system states.

But we now face the question: In the state (150), is the spin-up system identical to the spin-right or the spin-left system? The proponent of *Coalesce* must allow both possibilities, so far as probabilities are concerned, since quantum mechanics tells us that $p(\uparrow | \uparrow) = p(\rightarrow | \uparrow) = \frac{1}{2}$, etc.

Thus the proponent of *Coalesce* is forced to claim that the state (150)—and, with it, the entire quantum formalism—is incomplete as representation of the physics, since it does not give us complete information about the constituents of the assembly. Rather, the state (150) must be interpreted as representing a statistical ensemble of collections of particles of both kinds. Indeed, since there are continuum-many bases in which the state is manifestly non-GM-entangled, the state must represent, for the proponent of *Coalesce*, a statistical ensemble of continuum-many kinds of assembly.

This unwelcome result would be enough to reject *Coalesce*. But in fact matters are much worse. The problems facing any attempt to interpret quantum probabilities as epistemic are well known; and it can be seen that *Coalesce* runs afoul the Kochen and Specker (1967) no-go theorem for non-contextual hidden-variable theories.

The types of theories addressed by the Kochen-Specker theorem seek to attribute a definite truth-value to every ray in Hilbert space such that the quantum probabilities may be interpreted as arising from statistical ensembles of states corresponding to such attributions. This problem is equivalent to colouring the the entire unit $2(d-1)$-sphere in the projective Hilbert space $P(\mathcal{H})$ (where $d := \dim(\mathcal{H})$) in black and white so that, in any family of perpendicular points, all but one point is painted black. The theorem states that this cannot be done if $d \geq 3$.

Now consider a non-GMW-entangled $N$-particle state in which no single-system state is occupied more than once. For this state, the mathematical problem facing the proponent of *Coalesce* is to attribute one of $N$ particle labels to every ray in the $N$-dimensional subspace of $\mathcal{H}$ spanned by the constituent systems’ states. Again, this must be done in a way that is consistent with the quantum probabilities arising from averages over statistical ensembles of assembly states that correspond to such attributions. This problem is equivalent to colouring the entire unit $2(N-1)$-sphere with $N$ colours, so that in any family of perpendicular points each colour is used only once among the $N$ points. But if we label just one of the colours ‘white’ and the remaining $N - 1$ ‘black’ (consider them as shades of black, as it were), then it is clear that this problem can be solved only if Kochen and Specker’s problem can be solved—which it cannot, for $N \geq 3$.

The only case for which *Coalesce* escapes the no-go theorem is $N = 2$. But this should come as no consolation, since we were seeking a general solution to the basis arbitrariness problem. We must therefore look elsewhere.\(^\text{16}\)

\(^{16}\)It is perhaps surprising to note that *Coalesce* does not run afoul of Bell’s Theorem (1964)—despite
7.7 The ‘Multiply’ response

We could not solve the basis arbitrariness problem by stipulating that constituent systems from different bases are identical, so the natural next suggestion is to try the opposite: i.e. to say that any two constituent systems from rival bases are distinct.

Aside from the ontological extravagance of this response, one immediately wonders why it is that the systems from rival bases are always correlated in the same way, to accord with the quantum probabilities. (Why, for example, is there always one spin-left and one spin-right fermion whenever there is one spin-up and one spin-down?) In short, it seems we have a multitude of necessary connections between distinct existences.

Hume’s dictum, that there are no such connections,\(^\text{17}\) has recently been subjected to some serious criticism (MacBride (1999), Wilson (2008)). It may be argued, for example, that necessary connections between distinct existences are perfectly in order, so long as the objects in question are related in the right way, despite being distinct. Being related in the ‘right way’ might include: one of the objects composing the other (if one believed that composition was not a form of identity); or being related by a difference in logical form, like particular to universal.

Now all these cases provide interesting challenges to Hume’s dictum. They may even enforce a limitation on its application. But the current case is clearly not of this type: the systems in question are all taken to be particulars, and they are all taken to be wholly distinct. Why should it be that they always appear together as they do?

A response available to the proponent of Multiply, inspired by Lewis (1992), is that the connection between the particles may not be absolutely necessary, but only physically necessary. The claim is that, somewhere in the full expanses of logical space, the constituent systems appear without systems from other bases, and it is only in the worlds that obey quantum mechanics that they, due to physical necessity, appear together.

However, this response is vulnerable to a serious objection pertaining to the classical limit. In the classical limit, particles with definite locations are identified with particles with definite momenta: but how can two groups of wholly distinct objects suddenly become identical in the classical limit? Multiply entails a sudden change where we need a continuous transition. This, together with worries about ontological extravagance and Hume’s dictum, points to a natural solution.

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\(^{150}\) being the very state that Bell used in his proof. The reason is that Bell’s Theorem assumes that the full algebra of quantities on \(\mathbb{C}^4\) is available; whereas here we impose PI, which greatly restricts the available algebra. In particular, joint probabilities for outcomes for different directions of spin cannot meaningfully be calculated, since the individuation criteria for the two systems would not be orthogonal.

\(^{17}\)Hume (1740, Book I, Part III, §VI) wrote, ‘There is no object, which implies the existence of any other if we consider these objects in themselves.’
7.8 The ‘Overlap’ response

The solution, and the best hope at solving the basis arbitrariness problem, is to retain
the absolutely necessary connections between constituent systems from different bases,
but deny that the systems are wholly distinct. This overcomes the Humean problem,
since the particles now fall outside the domain of application of Hume’s dictum. It
quells concerns about ontological extravagance, since particles from different bases may
compose the same assembly, and are therefore jointly identical. Finally, it gets the
classical limit right: we may say that systems with a definite location and those with a
definite momentum go from partial overlap to total coincidence in the classical limit.

To illustrate the Overlap response, consider again the singlet state
\[
\frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) = \frac{1}{\sqrt{2}} (|\rightarrow\rangle \otimes |\leftarrow\rangle - |\leftarrow\rangle \otimes |\rightarrow\rangle). 
\] (151)

The advocate of Overlap agrees with Coalesce, and against Multiply, that the assembly
is in the same state no matter with what single-system basis it is described. But
Overlap disagrees, both with Coalesce that the constituent systems are identical, and
with Multiply that they are wholly distinct. Rather, they partly overlap, so that e.g. the
spin-up system is somehow composed of parts of the spin-left and spin-right systems,
and that the sum of the spin-up and spin-down systems is exactly identical to
the sum of the spin-left and spin-right systems (and every pair corresponding to every
other decomposition). Similarly, the Bloch sphere has continuum-many decompositions
into hemispheres; there remains just one sphere despite this because hemispheres from
different decompositions party overlap.

We might even propose the squared inner product of two states as a measure
of overlap for two systems which occupy those states. Since, e.g. \(|\langle \uparrow | \leftarrow \rangle|^2 = |\langle \uparrow | \rightarrow \rangle|^2 = \frac{1}{2}\),
we might say that the spin-up system is composed of exactly half of the spin-left and
spin-right systems. This measure has the desirable property that, given any single-
system state, its degree of overlap with all of the single-systems states in a given basis
sums to unity.

However, overlap is part-identity, and part-identity is identity of parts. What are
the parts of the putatively overlapping constituent systems from different bases? One
suggestion is that the parts are the spatial parts of the systems’ wavefunctions. But this
suggestion has the wrong results, since there are many even functions \( f(x) \) that overlap
some odd function \( g(x) \); yet their inner product, as defined by \( \int_{-\infty}^{\infty} f^*(x)g(x) \, dx \), is
always zero (if it exists).

Here we run into a conflict between the classical and (given the results of section
7.2) fermionic algebras of composition. Mereological composition, of the type required
by Overlap, has the mathematical structure of a Boolean algebra. But fermionic com-
position, like the system of propositions in quantum logic, has the structure of a non-
distributive orthomodular lattice. One cannot talk of “parts” without accepting the
associated background theory—classical mereology—in which parthood is defined. But
this background theory is incompatible with our theory of quantum objects. It may be said that what we discovered here is a “non-classical theory of part-hood”, but this string of words is no easier to understand than the name “non-classical theory of probability”, sometimes attributed to systems of functions on a state space that do not obey the Kolmogorov axioms.

8 Conclusion: why worry?

I have argued that we can take experimentally observed conformity with permutation-invariance in quantum mechanics as evidence for the conclusion that what is permuted in the formalism—the order of the factor Hilbert spaces, or the factor Hilbert space “labels”—has no physical significance, just as in electromagnetism we take the gauge-invariance of all observed quantities as evidence against the physical significance of the four-vector potential. The consequence of doing so is that some reform is necessary in the usual way of doing quantum mechanics.

In particular, we need to reform our ideas of: (i) how to extract the state of any constituent system of an assembly; and (ii) what formal properties of the joint state indicate entanglement in a physically meaningful sense. Accordingly, I have proposed: (i) a means of extracting states of constituent systems conducive to the demands of permutation-invariance, using a method I call ‘qualitative individuation’; and (ii) an alternative definition of entanglement, GMW-entanglement, for which non-separability is a necessary but not sufficient condition. Both of these proposals rely on the fact that certain regions, called ‘individuation blocks’, of the joint bosonic or fermionic Hilbert spaces are unitarily equivalent to joint Hilbert spaces associated with “distinguishable” systems, in which permutation-invariance is not imposed.

The physical upshot of these proposals for the discernibility of “indistinguishable” systems are: (i) that fermions are always absolutely discernible; and (ii) that bosons are usually absolutely discernible, but sometimes utterly indiscernible. A more unwelcome result is that joint states of fermions which, physically speaking, ought not to count as entangled, are aptly represented by subspaces of the joint Hilbert space. The physical interpretation of this result is that fermions cannot be said to compose their assemblies, at least in any normal sense of the word ‘compose’.

There is a tempting, simple response to the puzzles discussed above. Since they arise from an attempt to understand quantum mechanics as a theory about objects that can be composed into assemblies, why not use the puzzles as a premise in an argument for the misguidedness of that understanding?

It can’t be denied that this response engenders a certain relief at having cut the Gordian knot. But on second thoughts, doubt and confusion creep back. For an understanding of the single-system Hilbert space and its associated algebra of operators is normally our gateway to understanding the joint Hilbert space and its associated algebra. (It is what allows us, for example, to interpret the operator $(\sigma \otimes 1 + 1 \otimes \sigma)$ as
the total spin operator.) It hard to see that we can take these heuristics seriously once we have given up on the homely idea that an assembly is composed out of constituent systems.

However, there is an alternative physical picture, which gives prominence to the points or regions of space-time, and understands particles and their hitherto-composites as patterns in the properties assigned to these regions (see e.g. Wallace & Timpson 2010). This picture is more familiar from quantum field theory. It is noteworthy that we are led to it from considerations confined entirely to elementary quantum mechanics.

9 References


schemes in relativistic quantum field theory’, *Philosophy of Science* 68, pp. 111-133.


